Applications of Fourier Series in Classical Guitar Technique

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In elementary differential equations courses, the model of a (transversely) vibrating string is frequently used to motivate the one-dimensional wave equation and Fourier series. For musically-inclined students, the motivation can be strengthened by applying this model to techniques used by classical guitarists to vary tone quality. This note describes two such applications. The first application (harmonics) is well-known, eliciting nods of familiarity from guitar-playing students. The second application (*sol tasto* and *sol ponticello*) is less well-known outside the specific context of trained classical guitarists, but it is equally interesting, suggesting a surprisingly simple mathematical explanation for the technique where it is not immediately obvious that such an explanation exists.

The physical model of a string in the xy-plane, with ends fixed at the points $(0,0)$ and $(\pi,0)$, displaced to the shape of a curve $y=f(x)$ at time $t=0$ and then released to move in the y-direction only (see Figure 1), gives rise to the boundary-value problem

$$a^2\frac{\partial^2 y}{\partial x^2}=\frac{\partial^2 y}{\partial t^2}$$

$$y(0,t)=0, \quad y(\pi,t)=0, \quad \left.\frac{\partial y}{\partial t}\right|_{t=0}=0, \quad y(x,0)=f(x),$$

where the constant a^2 is a parameter representing the ratio of the tension T along the string to the linear mass density m of the string (see, for example, **[1]** for an informal derivation of the equation from the physical situation). Bernoulli's solution of the above problem,

$$y(x,t)=\sum_{n=1}^{\infty} a_n \sin(nx)\cos(nat),$$

leads immediately to the idea of Fourier series. Setting $t=0$ in the solution suggests that the shape of the string at time $t=0$ (assumed to be $f(x)$) can be written as a

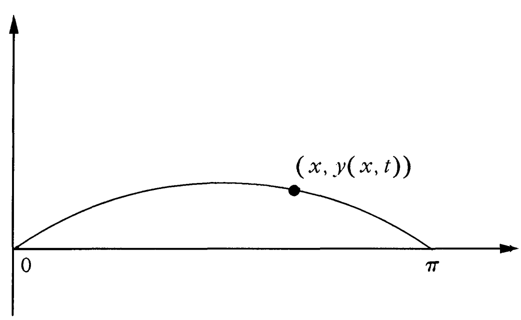

Figure 1

Fourier Sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx),$$

the coefficients a_n of which can be found explicitly in terms of $f(x)$:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

(by a well-known method that relies on the orthonormality of the set $\{\sin(nx)\}$; see **[1]** for details).

Guitarists (and other string instrumentalists) obtain soft, high-pitched tones using *harmonics*. Harmonics are best understood by first considering that Bernoulli's solution allows for only certain "modes of vibration" of the string—namely, the eigenfunctions $\sin(nx)$ for $n = 1, 2, \ldots$. Several of these modes are shown in Figure 2. Musically, each mode corresponds to a specific pitch (vibrational frequency). The mode for $n = 1$ corresponds to the lowest pitch produced by the string, and is called the *fundamental*. The other modes produce higher pitches, called *overtones*. The fundamental and overtones are collectively referred to as *partials*. Typically, a note produced by a musical instrument has the fundamental and a finite number of overtones audibly present; the relative strengths of these pitches (or *overtone pattern*) are what give an instrument its characteristic tone or timbre. Variations in tone are produced by manipulating the overtone pattern produced by the instrument. Harmonics are one such variation. To produce a harmonic, a string instrumentalist gently touches (damps) the string at a point known to be a fixed zero ("node") for some (but not all) of the partials (i.e., at a point π/n along a string of length π). Damping in this way removes all modes of vibration *except* those that have a node at the point touched. The resulting tone sounds softer because it is made up of fewer audible partials, and higher-pitched because the fundamental is among the partials removed.

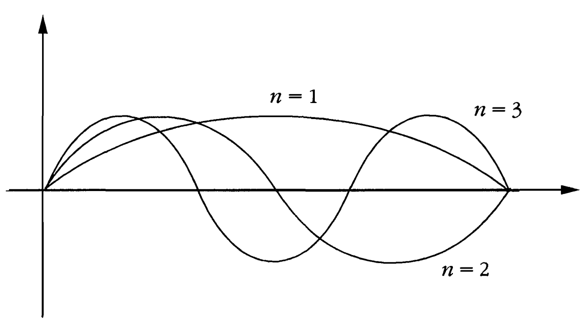

Figure 2

Classical guitarists also obtain variations in tone by varying the (horizontal) location at which the string is plucked. Specifically, plucking a string toward its center (*sol tasto* or "near the fingerboard") produces a soothing, mellow tone, and plucking a string close to one end (*sol ponticello* or "near the bridge") produces a biting, strident tone. It is an interesting exercise to compute relative strengths of partials as a function of plucking location. Such an analysis provides a plausible explanation for the differences between *sol tasto* and *sol ponticello* tone quality.

To perform the computations necessary for comparing relative strengths of partials, we need to have a realistic idea of what the initial shape function $f(x)$ is. For plucked-string instrument (of which the guitar is an example), the possible initial-shape functions can be modeled by the family

$$f_\alpha(x) = \begin{cases} \dfrac{\epsilon}{\alpha} x & 0 \le x \le \alpha \\ \dfrac{-\epsilon}{\pi - \alpha}(x - \pi) & \alpha < x \le \pi \end{cases}$$

where ϵ is a (small) constant representing the initial maximum vertical displacement, and α is the horizontal location at which the string is plucked (so $0 < \alpha < \pi$) (see Figure 3). The strength of the nth partial of a plucked string with initial shape $f_\alpha(x)$ is measured by the magnitude $|a_{n,\alpha}|$ of the nth Fourier coefficient of $f_\alpha(x)$. It is a short exercise using a computer algebra system (or a long but rewarding one by hand, involving integration by parts and lots of algebraic simplification) to show that

$$|a_{n,\alpha}| = \frac{2}{\pi}\left|\int_0^\pi f_\alpha(x)\sin(nx)\,dx\right| = \frac{2\epsilon|\sin(n\alpha)|}{n^2\alpha(\pi - \alpha)}.$$

Using this surprisingly compact expression, we can plot the graph of $|a_{n,\alpha}|$ as a function of α for $n = 1, 2, \ldots, 8$ (see Figure 4). The graph makes it obvious that the higher overtones diminish in strength relative to the fundamental and lower overtones as α approaches $\pi/2$ (*sol tasto*), and grow in relative strength as α approaches π (*sol ponticello*). This suggests that the *sol ponticello* tone is more biting simply because higher partials are more prevalent in the overtone pattern, and the *sol tasto* is more mellow because the lower partials are more prevalent.

An interesting applied mathematical project would be to test the above model empirically, using an actual guitar and sound analysis equipment sophisticated enough to measure relative strengths of individual overtones produced when a guitar string is plucked at various locations.

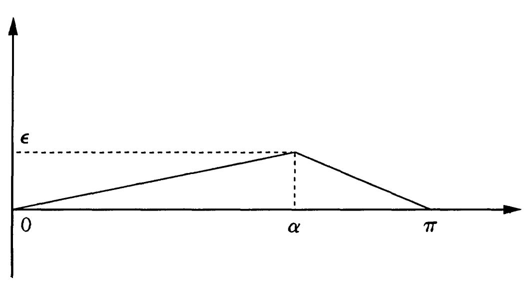

Figure 3

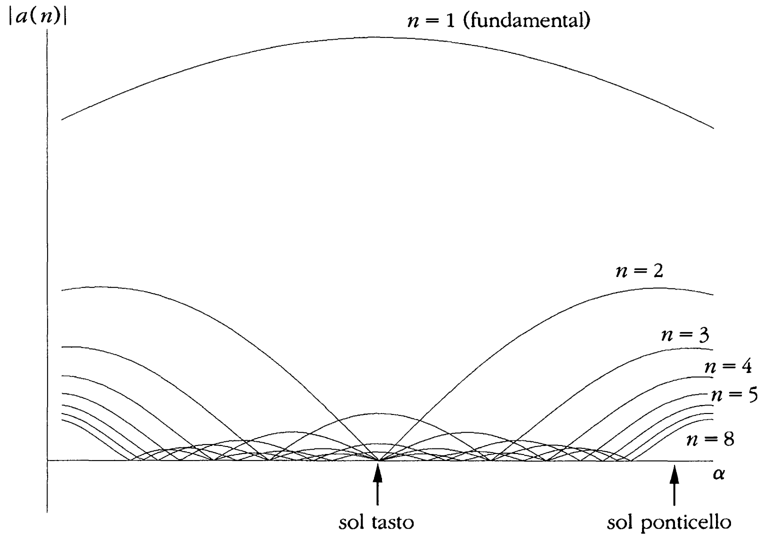

Figure 4

References

1. George F. Simmons, *Differential Equations with Applications and Historical Notes*, McGraw-Hill, Inc., 1972.

Fast-Food-Frusta and the Center of Gravity

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The fast-food-service container for one's favorite beverage also serves as a good max-min problem for calculus classes studying the center of gravity, and one for which the ensuing algebraic difficulties, nearly intractable by hand, can be easily managed with a computer algebra system (CAS). For a given container, the problem is to discover the liquid level for which the center of gravity is lowest. This level exists because as liquid is poured into an empty cup, the liquid lowers the center of gravity of the cup, but continuing to pour liquid into the cup eventually raises its center of gravity.

First, let us consider a cup in the shape of a cylinder of radius r, height H and mass m. Using a coordinate system with origin at the center of the base of the cup and vertical axis pointing upward, assume the center of gravity of the empty cup is $(0, h)$ with $h > 0$. Let y be the depth of a liquid of density δ in the cup; then $(0, \frac{y}{2})$ is the center of gravity of the liquid. We let $(0, f(y))$ be the center of gravity of the system—the cup together with its contents. See Figure 1a. This same problem appears as a *Soda Can Problem* in both **[1]** and **[2**, p. 812].