

0 or 1 (modulo 2) times, then this point may be designated as the root and the algorithm will produce a walk which visits every point the specified number of times.

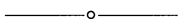
An *endline* of a tree T is a line incident with an endpoint of T , i.e., a point of degree one. The tree in Figure 2 has eight endpoints.

Theorem 2. *Given any endline of a tree T and $i \in \{0, 1, 2\}$, there exists a walk which originates at one of the points of the endline and visits every point i (modulo 3) times.*

If T has only two points, the result is immediate. Thus, assume T has at least three points. Let uv be an endline, where u is the endpoint. Embed T in the plane with v as root, u as the leftmost son of v and point w as the rightmost son. Now apply the algorithm of our Corollary. If all points, including v , are visited i (modulo 3) times, we are done. Otherwise, we may assume that the resulting walk W ends in w and visits all points i (modulo 3) times except for v . If v is visited $(i - 1)$ (modulo 3), then the number of times the walk Wv visits v is i (modulo 3) and we are done. Thus, it remains only to consider the case where the number of times W visits v is $(i + 1)$ (modulo 3). In this case, the walk $uvuWvu$ constructed from W visits all the points a correct number of times.

A slight modification of the preceding proof shows that of any two adjacent points in a tree, at least one can be used as the root of a tree for which the algorithm of our Corollary will produce a walk which visits each point of the tree i (modulo 3) times.

Habitué of video arcades may recognize the applicability of the preceding results to the game of “Q*Bert.”



A Note on Integration by Parts

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The point of a textbook exercise such as evaluating $\int x^n e^{ax} dx$ is to illustrate repeated integration by parts. Since this technique can be tedious for $n > 1$, students who have learned integration by parts may appreciate the following approach.

To evaluate $\int x^n e^{ax} dx$, assume that the answer is of the form $e^{ax}p(x)$, where $p(x)$ is a polynomial of degree n . Then obtain the coefficients of $p(x)$ by setting $D_x\{e^{ax}p(x)\}$ equal to $x^n e^{ax}$. This approach is not only simpler, it introduces students to a technique (the method of undetermined coefficients) they will encounter again in differential equations courses. We illustrate this approach as follows:

Example. To evaluate $\int x^3 e^{2x} dx$, we assume that an antiderivative of $x^3 e^{2x}$ is of the form $e^{2x}(Ax^3 + Bx^2 + Cx + D)$. Then

$$D_x \{ e^{2x} (Ax^3 + Bx^2 + Cx + D) \} = x^3 e^{2x}$$

yields

$$e^{2x} \{ 2Ax^3 + (3A + 2B)x^2 + (2B + 2C)x + (C + 2D) \} = x^3 e^{2x}.$$

This identity yields:

$$\begin{aligned}
2A &= 1 \\
3A + 2B &= 0 \\
2B + 2C &= 0 \\
C + 2D &= 0.
\end{aligned}$$

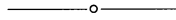
Since

$$A = 1/2, \quad B = -3/4, \quad C = 3/4, \quad D = -3/8,$$

we have

$$\int x^3 e^{2x} dx = \left(\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}\right)e^{2x} + K.$$

Other integrals, such as $\int e^{ax} \sin bx dx$, which require repeated integration by parts can also be evaluated more efficiently using this technique.



Relating Differentiability and Uniform Continuity

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For a continuous function $f: R \rightarrow R$ define function $F(x) = (f(x) - f(a))/(x - a)$ where a denotes some fixed real number. Clearly F is a continuous function defined on $I - a$, where I is any interval containing a . We wish to prove the following result.

The function $f(x)$ is differentiable at $x = a$ if and only if $F(x)$ is uniformly continuous on some punctured interval $I - a$.

If $f'(a)$ exists, then we may extend $F(x)$ to the continuous function

$$G(x) = \begin{cases} F(x), & x \neq a \\ f'(a), & x = a. \end{cases}$$

Since $G(x)$ is then uniformly continuous on any closed interval I containing a , it follows that $F(x)$ is uniformly continuous on $I - a$.

Suppose, conversely, that $F(x)$ is uniformly continuous on $I - a$, and let $\{x_n\} \in I - a$ be any sequence which converges to a . The sequence $\{x_n\}$ is then a Cauchy sequence and, since $F(x)$ is uniformly continuous, the sequence $\{F(x_n)\}$ is also a Cauchy sequence. By the completeness of the real numbers, there exists a number L such that $F(x_n) \rightarrow L$. Furthermore, L does not depend on the choice of $x_n \rightarrow a$. Indeed, suppose $\{y_n\} \in I - a$ is another sequence converging to a and let $F(y_n)$ converge to L' . Then the sequence $\{z_n\} \in I - a$ defined by

$$z_n = \begin{cases} x_{(n+1)/2}, & \text{if } n \text{ is odd} \\ y_{n/2}, & \text{if } n \text{ is even} \end{cases}$$

also converges to a , and $L = \lim_{n \rightarrow \infty} F(x_n) = \lim_{n \rightarrow \infty} F(z_n) = \lim_{n \rightarrow \infty} F(y_n) = L'$. This independence of L on the choice of $x_n \rightarrow a$ means that $L = \lim_{x \rightarrow a} F(x) = \lim_{x \rightarrow a} (f(x) - f(a))/(x - a) = f'(a)$.

