Algorithms for Evaluation of Polynomials

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Two efficient algorithms for evaluation of polynomials are Horner's method and synthetic division. Unfortunately, students do not seem to know that these methods are equivalent. In this note, we demonstrate their equivalence, and also show how Horner's method is implemented on a hand calculator.

Horner's method. Horner's method is based on expressing polynomials in a nested factored form, for example writing $P(x) = 4x^3 + 7x^2 - 2x + 1$ as $x[x(4x+7)-2]+1$, $Q(x) = 2x^4 - 7x^3 - 19x^2 + 8x - 3$ as $x\{x[x(2x-7)-19]+8\}-3$, etc. This form is especially suited to evaluation of polynomials. For instance, suppose we want to evaluate $P(x) = 4x^3 + 7x^2 - 2x + 1$ at $x = a$. We first express $P(x)$ in nested factored form as shown above. Then, starting from the innermost parentheses and working outward, we generate a sequence of values the last of which is $P(a)$:

$$\begin{aligned} P_0(a) &= 4, \\ P_1(a) &= 4a + 7 &&= aP_0(a) + 7, \\ P_2(a) &= a(4a+7) - 2 &&= aP_1(a) - 2, \\ P_3(a) &= a[a(4a+7)-2] + 1 = aP_2(a) + 1 = P(a). \end{aligned}$$

Translated into a sequence of computer or calculator steps, this procedure yields algorithms for evaluation of polynomials. Moreover, these algorithms are quite efficient: they require only n multiplications for polynomials of degree n.

Let us illustrate the calculator algorithm (assuming algebraic logic) with $P(x) = 4x^3 + 7x^2 - 2x + 1$. To calculate $P(a)$, we first enter the value a into the memory. (Then whenever we hit the recall button [RM] we get a, so effectively [RM] $= a$.) Next we key in the sequence

[4] [×] [RM] [+] [7] [×] [RM] [−] [2] [×] [RM] [+] [1] [=]

The result is $a[a(4a+7)-2]+1 = P(a)$.

Teaching note: Inexperienced students might expect the above key sequence to give $4a + 7a - 2a + 1 = 9a + 1$. It is important that they understand why it does not. In fact, I have considered approaching the whole topic of polynomial evaluation by asking the class to find the mistake in this "calculation of $9a + 1$." If they examine what the calculator actually does, they may discover nested factorization of polynomials for themselves.

Synthetic division. According to the remainder theorem, the value of a polynomial $Q(x)$ for $x = a$ equals the remainder when $Q(x)$ is divided by $x - a$. Synthetic division is a compact version of the long division algorithm for divisors of type $x - a$. To illustrate, let us evaluate $Q(x) = 2x^4 - 7x^3 - 19x^2 + 8x - 3$ at $x = 5$. Dividing $Q(x)$ by $x - 5$, the synthetic division looks like this:

5	2	−7	−19	8	−3
		10	15	−20	−60
	2	3	−4	−12	−63

First, we bring down the leading coefficient 2. Then we successively multiply the sum of each column by 5 and enter the product in the next column to the right.

The bottom line is that

$$Q(x) = 2x^4 - 7x^3 - 19x^2 + 8x - 3 = (x-5)(2x^3 + 3x^2 - 4x - 12) - 63.$$

The remainder is -63, the last number on the bottom. Hence, $Q(5) = -63$.

Equivalence of the methods. Rather than giving a general proof, let us illustrate the equivalence of the two methods for our polynomial $P(x) = 4x^3 + 7x^2 - 2x + 1$. It will be clear that the argument holds in general. To evaluate $P(a)$, we apply the synthetic division algorithm and see what happens:

$$\begin{array}{c|cccc} a & 4 & 7 & -2 & 1 \\ & & 4a & a(4a+7) & a[a(4a+7)-2] \\ \hline & 4 & 4a+7 & a(4a+7)-2 & a[a(4a+7)-2]+1 \end{array}$$

Thus the algorithm generates a sequence of values

$$P_0(a) = 4, \quad P_1(a) = aP_0(a) + 7, \quad P_2(a) = aP_1(a) - 2,$$
$$P_3(a) = aP_2(a) + 1 = P(a),$$

the same sequence Horner's method does, and generated in the same way. Thus the two algorithms are equivalent because they do exactly the same thing.

References

1. A. Flores, Computer-calculated roots of polynomials, *The Ideas of Algebra, K-12*, 1988 Yearbook of the National Council of Teachers of Mathematics, NCTM, Reston, Virginia, pp. 164–169.
2. C. Philips and B. Cornelius, *Computational Numerical Methods*, Ellis Harwood, 1986, pp. 67–68.
3. J. J. Price and H. Flanders, *College Algebra*, Saunders, Philadelphia, 1982.

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"I call the proof Euclidean because it entirely obscures the truth."

Professor Peter Borwein, Dalhousie University, 1989