Both a Borrower and a Lender Be

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It is well known that an initial amount P_0, earning an annual interest rate r (expressed as a decimal) compounded m (equally spaced) times per year, will grow after t years to a principal balance

$$P(t) = P_0(1 + r/m)^{mt}.$$

Most calculus texts prove that

$$P(t) = P_0 e^{rt}$$

in the limit as m approaches infinity. Accounts yielding according to this formula are said to be *compounded continuously*. This last equation is also obtained by solving the differential equation

$$\frac{dP}{dt} = rP, \qquad P(0) = P_0.$$

What most calculus texts fail to point out is that the repayment of a simple interest loan leads to similar, but more general results.

In simple interest loans (the industry standard), each payment is used first to pay the interest having accumulated since the previous payment, and then to reduce the principal balance. Suppose that in a simple interest loan, an amount P_0 is borrowed at an annual interest rate r, and that each year a total amount k is paid to the lender in m equal (and equally spaced) installments. (Thus, each payment is k/m.) Then after the first payment, the principal balance is

$$P_1 = P_0 - (k/m - (r/m)P_0) = P_0(1 + r/m) - k/m,$$

since $k/m - (r/m)P_0$ is the amount by which the payment exceeds the interest due. By induction, we find that the outstanding principal after the nth payment is

$$P_n = P_0(1 + r/m)^n - (k/m)\sum_{j=0}^{n-1}(1 + r/m)^j.$$

Upon summing the above geometric series and putting $n = mt$, we find that the principal balance after t years is

$$P(t) = P_0(1 + r/m)^{mt} - (k/r)\big((1 + r/m)^{mt} - 1\big).$$

In the limit as m approaches infinity, this becomes

$$P(t) = P_0 e^{rt} - (k/r)(e^{rt} - 1) = k/r - (k/r - P_0)e^{rt}.$$

Note that this equation is the solution to the differential equation

$$\frac{dP}{dt} = rP - k, \qquad P(0) = P_0.$$