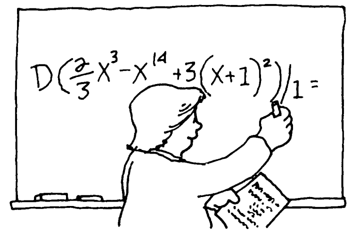
CLASSROOM CAPSULES

EDITOR

Nazanin Azarnia
Department of Mathematics
Santa Fe Community College
Gainesville, FL 32606-6200



A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Nazanin Azarnia.

An Archimedean Property of the Bicylinder

Duane W. DeTemple, Washington State University, Pullman, WA 99164-3113

In his work *On the Sphere and the Cylinder*, Archimedes showed that the volume of the sphere is exactly two thirds of the volume of the circumscribed cylinder, and the area of the sphere is also exactly two thirds of the area of the circumscribed cylinder. Archimedes was so pleased with these discoveries that he requested that a figure showing a sphere and its circumscribed cylinder (like Figure 1) be engraved on his tombstone.

An analogous property holds for the bicylinder and its circumscribing cube. A bicylinder is the surface formed by the intersection of two cylinders of equal radius whose axes meet orthogonally. Figure 2 shows a bicylinder and its circumscribing cube, as rendered by a *Mathematica* 3D plot. If the intersecting cylinders have radius r it is evident that the circumscribing cube has area $24r^2$ and volume $8r^3$.

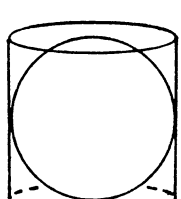


Figure 1
A sphere with its circumscribed cylinder.

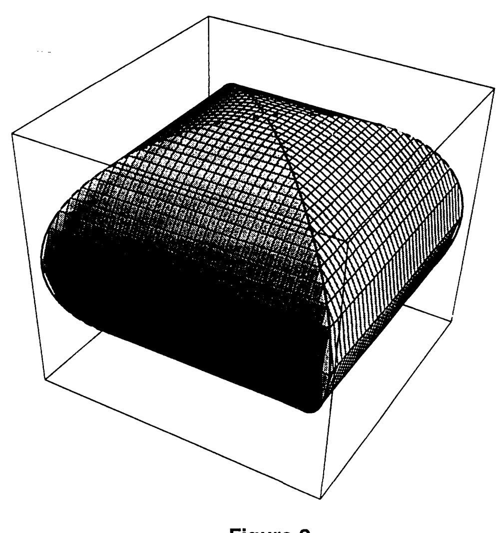


Figure 2
A bicylinder and its circumscribing cube.

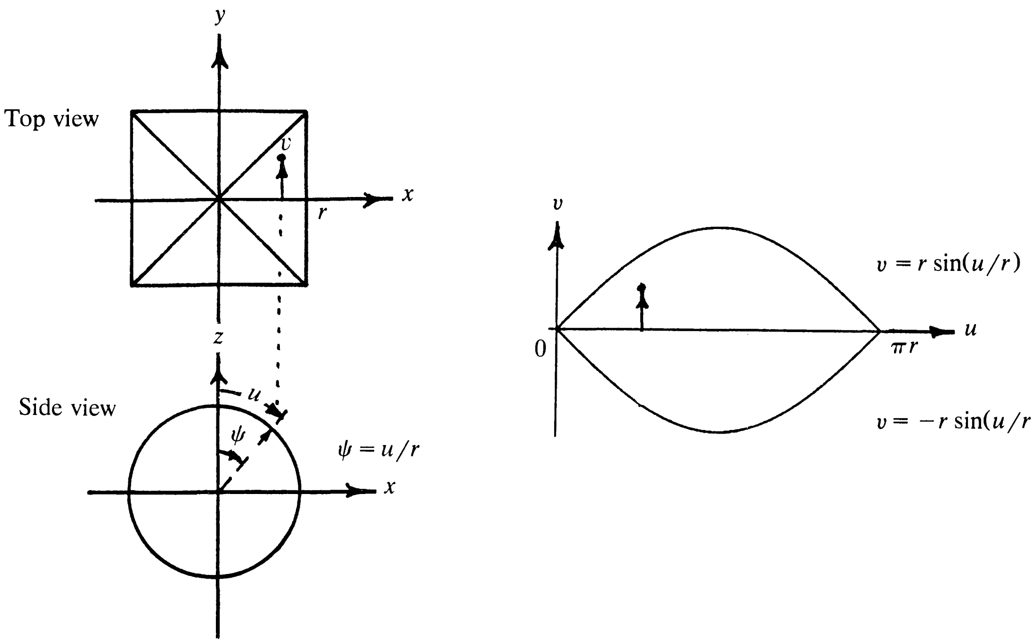


Figure 3
Each panel of the bicylinder unrolls to form a sinusoidal lens-shaped region.

To calculate the surface area of the bicylinder, it is convenient to assume that the axes of the intersecting cylinders coincide with the x- and y-axes. The top and front views of the bicylinder in Figure 3 show that the panel cut by the positive x-axis is given by the parametric equations

$$x = r\sin(u/r), \qquad y = v, \qquad z = r\cos(u/r),$$

where

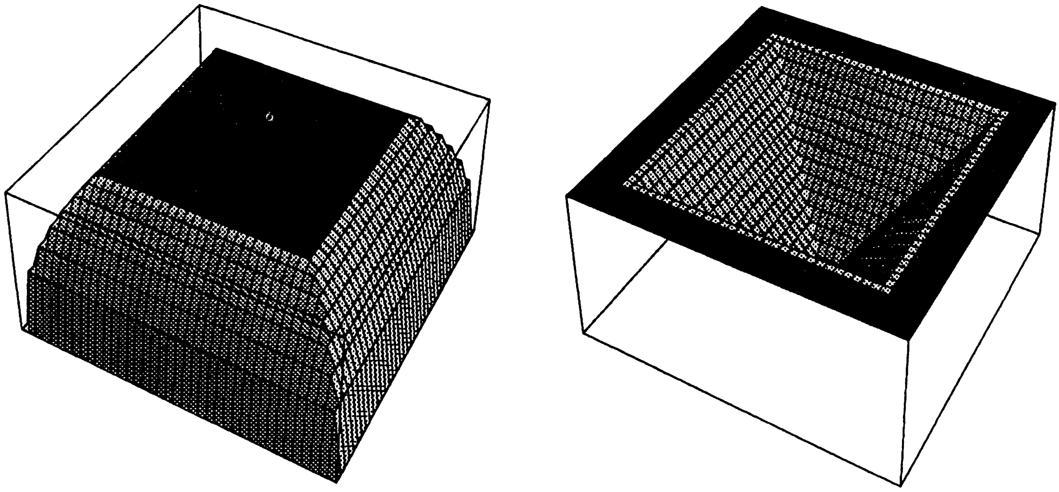
$$0 \le u \le \pi r \qquad \text{and} \qquad -r\sin(u/r) \le v \le \sin(u/r).$$

This shows that the panel can be unrolled to form the sinusoidal lens shown at the right of Figure 3. Each lens has area $4r^2$, since the upper half of the lens has area

$$\int_0^{\pi r} r\sin(u/r)\,du = -r^2\cos(u/r)\big|_0^{\pi r} = 2r^2.$$

Therefore the area of the bicylinder is $16r^2$ and the ratio of this area to that of the circumscribing cube is $16r^2/24r^2 = 2/3$. (For an interesting project, four large sinusoidal lenses can be photocopied onto cardstock; cutting and rolling the four lenses form a nice paper model of the bicylinder. To join the adjacent edges, place triangular tabs that can be glued and tucked away inside the surface.)

The volume of the bicylinder can be computed directly, but it is more interesting to compare volumes by means of Cavalieri's Principle. In Figure 4a the upper half $z \ge 0$ of the bicylinder is intersected by a horizontal plane at height z. The plane intersects the bicylindric solid in a square region (shown black) of side length $2(r^2 - z^2)^{1/2}$. Thus the square has area $4(r^2 - z^2)$. Figure 4b shows the upper half of the circumscribing cube, from which a square-based pyramid has been removed.



a. b.

Figure 4
The black square and the black square-shaped ring cut by a plane at height z have the same area.

The horizontal plane at height z intersects this solid in a square-shaped ring of area $(2r)^2 - (2z)^2$. Since both of the regions cut by the plane have the same area, namely $4r^2 - 4z^2$, the two solids have equal volume by Cavalieri's Principle. The pyramid has one third of the volume of the corresponding rectangular solid, so we conclude that the volume of the bicylinder is two thirds that of its circumscribing cube.

Can anyone refer me to a good tombstone engraver? No hurry, of course.

Round-off, Batting Averages, and Ill-Conditioning

Edward Rozema, University of Tennessee, Chattanooga, TN 37403-2598

One summer day, as I was watching a televised Atlanta Braves baseball game, Mark Lemke, the Braves' second baseman, got 5 hits in 6 official at bats.* The television announcer, Skip Carey, commented that Lemke raised his batting average from .182 to .210; however, he didn't give Lemke's total number of hits or his total number of official at bats. This raised a question which I thought would be suitable for my precalculus classes: How many total hits and official at bats did Lemke have at the beginning of the game? This simple question will throw us a few curves and lead to an investigation of round-off errors, ill-conditioned systems, and interval analysis.

*For this article, you will need to know that a player's *batting average* is computed by dividing the total number of hits by the total number of official at bats (both are integers). The batting average is always reported after rounding the value to three correct decimal digits. For example, if a player has 20 hits in 70 official at bats, then the batting average is $20/70 = 0.286$.