Now consider the case of $r = -m/n$, where m and n are positive integers. Here

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{-m/n} - x^{-m/n}}{h} = \lim_{h\to 0} \frac{\left\{(1/(x+h))^{1/n}\right\}^m - \left\{(1/x)^{1/n}\right\}^m}{h},$$

which can be recast as (3):

$$f'(x) = \lim_{h\to 0} \frac{x^{m/n} - (x+h)^{m/n}}{h(x+h)^{m/n}x^{m/n}} = \lim_{h\to 0} \frac{-1}{(x+h)^{m/n}x^{m/n}} \cdot \lim_{h\to 0} \frac{(x+h)^{m/n} - x^{m/n}}{h}.$$

By definition, the last limiting quotient in (3) is the right-hand expression in (2). Hence,

$$f'(x) = \lim_{h\to 0} \frac{-1}{(x+h)^{m/n} \cdot x^{m/n}} \cdot (m/n)x^{(m/n)-1} = -(m/n)x^{-(m/n)-1}.$$

Average Values and Linear Functions

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The mean value $M(f;\, a,b) = \frac{1}{b-a}\int_a^b f(t)\,dt$ of an integrable, real-valued function f defined on the interval $[a,b]$ arises frequently in elementary calculus. For example, when f is the continuous instantaneous rate of change (the derivative) of a function g, then (by the fundamental theorem of calculus)

$$M(f;\, a,b) = \frac{1}{b-a}\int_a^b g'(t)\,dt = \frac{g(b)-g(a)}{b-a}$$

is just the average rate of change of g over $[a,b]$. Using the mean value theorem for integrals, one can also show for continuous f that $M(f;\, a,b) = f(c)$ for at least one value of $c \in (a,b)$. As a third illustration, one can express the definite integral of f as the limit of Riemann sums

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(e_i) \cdot \left(\frac{b-a}{n}\right), \qquad e_i = a + i\left(\frac{b-a}{n}\right)$$

and observe that

$$M(f;\, a,b) = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{i=1}^{n} f(e_i);$$

the mean of the function is a limit of "discrete" means.

The purpose of this note is to demonstrate how the mean value of an integrable function can be used to characterize the function's linearity. Quite apart from their intrinsic interest, the arguments below should be helpful to instructors seeking enrichment material that reinforces many of the topics studied in calculus.

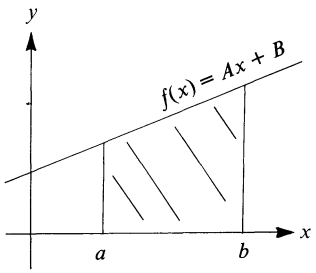

Figure 1.

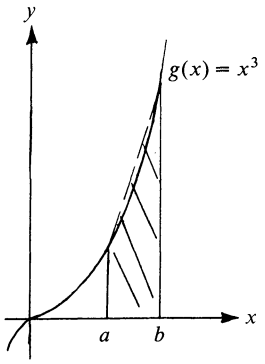

Figure 2.

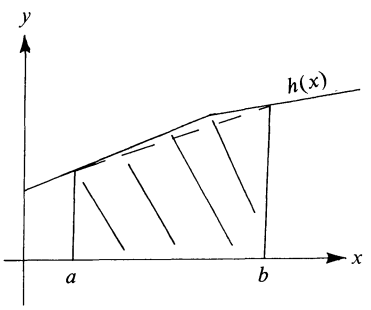

Figure 3.

For a linear function $f(x) = Ax + B$, the value $(b - a)M(f;\ a,b)$ is the area of the shaded trapezoid (Figure 1) with height $b - a$ and bases of length $f(a)$, $f(b)$. Thus,

$$M(f;\ a,b) = \frac{f(a) + f(b)}{2} \qquad \text{for all} \quad a,b \in \mathbb{R}. \tag{1}$$

As Figures 2 and 3 illustrate, (1) fails for the nonlinear functions g, h. Thus, it seems reasonable to attempt to characterize linearity in terms of (1).

Theorem 1. *A real-valued function f on $[a,b]$ is linear if and only if f is continuous and $M(f;\ a,x) = \dfrac{f(a) + f(x)}{2}$ for all $x \in (a,b)$.*

Proof. A linear function clearly satisfies the asserted conditions. Suppose, conversely, that f is continuous and satisfies $M(f;\ a,x) = \dfrac{f(a) + f(x)}{2}$ for all $x \in (a,b)$. Expressing this equality as

$$2\int_a^x f(t)\,dt = (x - a)\{f(a) + f(x)\} \qquad x \in (a,b) \tag{2}$$

and differentiating both sides of (2) twice, we obtain

$$2f'(x) = (x - a)f''(x) + f'(x) + f'(x).$$

Thus, $(x - a)f''(x) = 0$ for all $x \in (a,b)$. This, of course, means that $f'(x)$ is a constant and $f(x)$ is linear for all $x \in (a,b)$. Because f is continuous on $[a,b]$, it follows that f is linear on all of $[a,b]$.

It is straightforward to verify that a linear function f satisfies

$$\frac{f(a) + f(b)}{2} = f\left(\frac{a + b}{2}\right) \qquad \text{for all} \quad a,b \in \mathbb{R}. \tag{3}$$

This property can also be used to characterize a function's linearity.

Theorem 2. *A real-valued function f on $[a,b]$ is linear if and only if f is differentiable on $[a,b)$ and continuous at b, f' is continuous at a, and $f\left(\frac{a+x}{2}\right) = \frac{f(a)+f(x)}{2}$ for all $x \in (a,b)$.*

Proof. We assume that f has the asserted properties and show that f is linear. Begin (using the chain rule) by differentiating $f\left(\frac{a+x}{2}\right) = \frac{f(a)+f(x)}{2}$ to obtain

$$f'(x) = f'\left(\frac{a+x}{2}\right). \tag{4}$$

Replacing x with $(a+x)/2$ in (4), we get

$$f'\left(\frac{a+x}{2}\right) = f'\left[\frac{a+\left(\frac{a+x}{2}\right)}{2}\right].$$

Iteration produces a sequence

$$x_1 = \frac{a+x}{2} > x_2 = \frac{a+\left(\frac{a+x}{2}\right)}{2} > x_3 > \cdots > x_n = \frac{(2^n-1)a+x}{2^n} > \cdots$$

such that $f'(x) = f'(x_n)$ for each n and $\lim_{n\to\infty} x_n = a$. Since f' is continuous at a, we have $f'(a) = \lim_{n\to\infty} f'(x_n) = f'(x)$. In other words, $f'(x)$ is the constant value $f'(a)$ for all $x \in (a,b)$. Therefore, f is linear on (a,b). Since f is continuous at a and b, it follows that f is linear on $[a,b]$.

For a linear function f, properties (1) and (3) yield

$$M(f;\, a,b) = f\left(\frac{a+b}{2}\right) \qquad \text{for all} \quad a,b \in \mathbb{R}. \tag{5}$$

Therefore, we offer another characterization of linearity in terms of (5).

Theorem 3. *A real-valued function f defined on an open interval I is linear if and only if f'' is continuous on I and $M(f;\, a,x) = f\left(\frac{a+x}{2}\right)$ for all $a,x \in I$ with $x > a$.*

Proof. A function f meeting the asserted conditions must have $f'' = 0$ on I. Suppose not; that is, suppose without loss of generality that $f''(b) > 0$ for some $b \in I$. Then (since f'' is continuous) $f''(c) > 0$ for all c in some open subinterval J of I. Therefore, f' is strictly increasing on J. Now consider $a,x \in J$ with $a < x$. By clearing denominators in $M(f;\, a,x) = f\left(\frac{a+x}{2}\right)$, differentiating, and rearranging algebraically, one readily obtains

$$\left(\frac{x-a}{2}\right)\cdot f'\left(\frac{a+x}{2}\right) = f(x) - f\left(\frac{a+x}{2}\right). \tag{6}$$

However, by the Mean Value theorem, the right-hand side of (6) equals $f'(d)\cdot\left(\frac{x-a}{2}\right)$ for some $d \in J$ strictly between $\frac{a+x}{2}$ and x. But this means

$$f'\left(\frac{a+x}{2}\right) = f'(d), \tag{7}$$

contradicting the fact that f' is strictly increasing on J. Our proof that $f'' = 0$ on I is thus complete.

Remarks. (a) For a proof of Theorem 3, emphasizing the continuity of f'', assume that $[a, x] \subset I$ and reason as above to obtain (7). Then, by the Mean Value theorem (or Rolle's theorem), $f''(x_1) = 0$ for some x_1 strictly between $(a + x)/2$ and x. Replace a in the preceding argument by x_1, thus obtaining an x_2 such that

$$\frac{\frac{a+x}{2} + x}{2} < \frac{x_1 + x}{2} < x_2 < x$$

and $f''(x_2) = 0$. By iteration, we obtain a sequence $x_1 < x_2 < x_3 < \cdots$ such that all $f''(x_n) = 0$ and $\lim_{n\to\infty} x_n = x$. The continuity of f'' at $x \in I$ therefore yields $f''(x) = \lim_{n\to\infty} f''(x_n) = 0$.

(b) We state without proof the following companion for Theorem 3. *Let a real-valued function f be given on an open interval I containing* 0 *by a convergent power series $\sum a_n x^n$. Then f is linear if and only if I contains a positive number r such that $M(f;\, 0, x) = f\left(\frac{x}{2}\right)$ for all $x \in (0, r)$.*

On Rearrangements of the Alternating Harmonic Series

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The two series most familiar to beginning calculus students are the Harmonic Series (usually a student's first example of a divergent series whose terms approach zero) and the Alternating Harmonic Series (the first conditionally convergent series). When Taylor series are studied, it is shown that the Alternating Harmonic Series (abbreviated AHS) actually converges to $\ln 2$.

Because of its familiarity, the AHS is a reasonable candidate for illustrating how conditionally convergent series may be rearranged to change their sums. For example, we may replace each odd term x of the AHS by $(2x - x)$ and get a pattern in which one positive term is followed by two negative terms. If we then multiply each term of the new series by $1/2$, we get a rearrangement of the AHS which converges to half of the original sum. Thus,

$$\begin{aligned}\ln 2 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots \\ &= 2 - 1 - \tfrac{1}{2} + \tfrac{2}{3} - \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{2}{5} - \tfrac{1}{5} - \tfrac{1}{6} + \cdots,\end{aligned}$$

and the rearranged AHS satisfies

$$(\tfrac{1}{2})\ln 2 = 1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{5} - \tfrac{1}{10} - \cdots.$$

This is an example of a *regular rearrangement*, in which there is a regular pattern consisting of a fixed number of positive terms taken in order, followed by a fixed number of negative terms taken in order. We use $A(m, n)$ to denote such an ordered rearrangement consisting of *m positive terms* followed by *n negative terms*. Thus, the