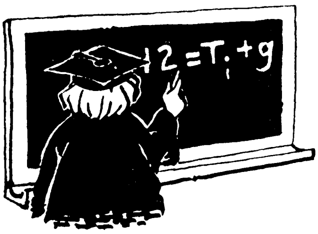

CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

Readers are invited to submit material for consideration to:

Warren Page
Department of Mathematics
New York City Technical College
300 Jay Street
Brooklyn, N.Y. 11201

Three Ways to Maximize the Area of an Inscribed Quadrilateral

Leroy F. Meyers, The Ohio State University, Columbus, OH

Among all quadrilaterals inscribed in a circle, the square has the largest area.

A geometric proof of the above theorem (Figure 1) was given by Peter Renz [TYCMJ 11 (March 1980) 147–149] in response to a trigonometric proof (Figure 2) by Ivan Niven [TYCMJ 10 (June 1979) 162–168, generalized to n-gons in *Maxima and Minima without Calculus*, MAA Dolciani Mathematical Expositions, No. 6 (1981) 117–118] based on the inequality

$$\sum_{i=1}^{n} \sin\alpha_i \leqslant n\sin\left(\frac{1}{n}\cdot\sum_{i=1}^{n}\alpha_i\right) \qquad \text{for} \quad \alpha \in [0,\pi], \tag{1}$$

with equality holding if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n$.

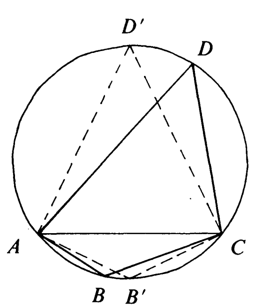

Figure 1.

If $ABCD$ is not a square, we can construct a quadrilateral $AB'CD'$ whose area is larger than that of $ABCD$ by using the fact that the triangle of largest area with given base and given opposite angle is isosceles.

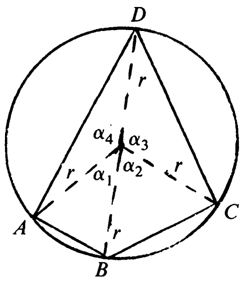

Figure 2.

By formula (1), we have

$$K = \tfrac{1}{2}r^2(\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3 + \sin\alpha_4)$$
$$\leqslant \tfrac{1}{2}r^2 \cdot 4\sin(\pi/2)$$
$$= 2r^2.$$

A third proof, given below, is presented in order to illustrate how one problem can have solutions proceeding along several different lines, each solution providing different insights and possibilities for generalization. This proof is based on the fact that the area K of any plane quadrilateral $ABCD$ is given by

$$K = \tfrac{1}{2}|AC|\,|BD|\sin\alpha = \tfrac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|,$$

where α is either angle between the diagonals AC and BD, produced if necessary. The formula is valid even if $ABCD$ cannot be inscribed in a circle (Figure 3a) or is nonconvex (Figure 3b). In fact, if BD is an internal diagonal of $ABCD$ (there is at least one such) and the diagonals meet at P, then the area of $ABCD$ is the sum of the areas of triangles ABD and CBD:

$$K = \tfrac{1}{2}|BD|\,|PA|\sin\alpha + \tfrac{1}{2}|BD|\,|PC|\sin\alpha$$
$$= \tfrac{1}{2}|BD|(|PA| + |PC|)\sin\alpha = \tfrac{1}{2}|BD|\,|AC|\sin\alpha.$$

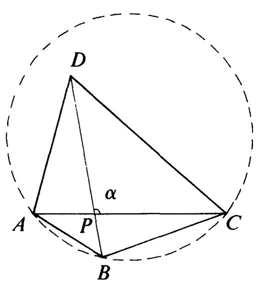

Figure 3a. Convex quadrilateral.

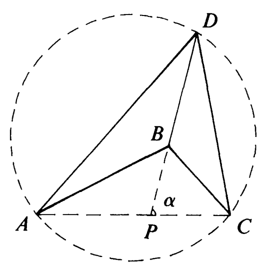

Figure 3b. Nonconvex quadrilateral.

Now consider any quadrilateral $ABCD$ inscribed in a circle of radius r. Since the diagonals of an inscribed quadrilateral are not longer than the diameter of the circle, we have

$$K = \tfrac{1}{2}|AC|\,|BD|\sin\alpha \leqslant \tfrac{1}{2} \cdot 2r \cdot 2r \cdot 1 = 2r^2,$$

with equality holding just when $|AC| = |BD| = 2r$ and $\alpha = \pi/2$. But an inscribed quadrilateral has perpendicular diameters as diagonals if and only if the quadrilateral is a square.

It may be noted that this also proves that the area of an inscribed triangle is less than the area of the inscribed square, since the triangle may be considered a degenerate quadrilateral, with $C = D$.