An Alternate Proof of the Vector Triple Product Formula

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Here is a proof of the vector triple product formula

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \tag{1}$$

which is based on the scalar triple product formula

$$[\mathbf{bcn}] = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{n}) = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{n} \tag{2}$$

and the observation that

$$\mathbf{n} \times (\mathbf{c} \times \mathbf{n}) = |\mathbf{n}|^2\mathbf{c} \qquad \text{for} \quad \mathbf{c} \perp \mathbf{n}. \tag{3}$$

(Note that $\mathbf{c} \perp \mathbf{n}$ implies $|\mathbf{n} \times (\mathbf{c} \times \mathbf{n})| = |\mathbf{n}|^2|\mathbf{c}|$ and, by the right-hand rule, $\mathbf{n} \times (\mathbf{c} \times \mathbf{n})$ points in the same direction as $\mathbf{c}$.)

The proof of (1) is trivial if $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent. Assume they are not, in which case $\mathbf{b} \times \mathbf{c} = \mathbf{n} \neq 0$ and the perpendicularity property of the cross product implies that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is in the plane of $\mathbf{b}$ and $\mathbf{c}$. Thus,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \lambda\mathbf{b} + \mu\mathbf{c} \tag{4}$$

for some scalars λ, μ. To determine λ and μ, take the dot product of both sides with $(\mathbf{c} \times \mathbf{n})/[\mathbf{bcn}]$ (where $\mathbf{n} = \mathbf{b} \times \mathbf{c}$) and note that $[\mathbf{bcn}] = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{n} = |\mathbf{n}|^2$. This yields

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \cdot \frac{(\mathbf{c} \times \mathbf{n})}{[\mathbf{bcn}]} = \lambda \frac{\mathbf{b} \cdot (\mathbf{c} \times \mathbf{n})}{[\mathbf{bcn}]} + \mu \frac{\mathbf{c} \cdot (\mathbf{c} \times \mathbf{n})}{[\mathbf{bcn}]}$$

$$\frac{(\mathbf{a} \times \mathbf{n}) \cdot (\mathbf{c} \times \mathbf{n})}{|\mathbf{n}|^2} = \lambda \cdot 1 + \mu \cdot 0$$

$$\frac{\mathbf{a} \cdot (\mathbf{n} \times (\mathbf{c} \times \mathbf{n}))}{|\mathbf{n}|^2} = \lambda$$

$$\frac{\mathbf{a} \cdot |\mathbf{n}|^2\mathbf{c}}{|\mathbf{n}|^2} = \lambda \qquad [\text{via } (3)]$$

$$\mathbf{a} \cdot \mathbf{c} = \lambda.$$

Observe that, since $\mathbf{c} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{c})$,

$$\mathbf{a} \times (\mathbf{c} \times \mathbf{b}) = -\mu\mathbf{c} - \lambda\mathbf{b}.$$

Now begin with (4), and proceed anew with $\mathbf{b}$ and $\mathbf{c}$ interchanged. This yields

$$\mathbf{a} \cdot \mathbf{b} = -\mu, \tag{5}$$

completing the proof of (1). Alternatively, one may obtain (5) by taking the dot product of (4) with $(\mathbf{b} \times \mathbf{n})/[\mathbf{bcn}]$.

It is possible to make the entire proof "thumb free" by proving (3) as follows. The perpendicularity property implies

$$\mathbf{n} \times (\mathbf{c} \times \mathbf{n}) = \alpha\mathbf{c} + \beta\mathbf{n}. \tag{6}$$

Taking the dot product of (6) with $\mathbf{n}$ gives $0 = \alpha\mathbf{0} + \beta\mathbf{n} \cdot \mathbf{n}$, yielding $\beta = 0$. Taking the dot product with **c** and using (2) yields

$$\alpha\mathbf{c}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{n}\times(\mathbf{c}\times\mathbf{n}) = (\mathbf{c}\times\mathbf{n})\cdot(\mathbf{c}\times\mathbf{n}) = |\mathbf{c}\times\mathbf{n}|^2 = |\mathbf{c}|^2|\mathbf{n}|^2;$$

so $\alpha = |\mathbf{n}|^2$, proving (3). This shows that (3), and thus (1), would remain valid if the cross product were defined by a left-hand rule instead of a right-hand rule.

A Nonstandard Solution to a Standard Problem

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Sometimes, the usual solution to a standard problem becomes so routine that one never thinks to look beyond it for an alternate solution. A case in point is the old standby:

> *Find the equation of the circle, given the coordinates $P_1(a_1,b_1)$ and $P_2(a_2,b_2)$ of the endpoints of a diameter.*

The standard approach leads us to locate the center, calculate the radius, and then substitute into the formula for the equation of a circle to obtain

$$\left(x - \frac{a_1+a_2}{2}\right)^2 + \left(y - \frac{b_1+b_2}{2}\right)^2 = \frac{(a_1-a_2)^2+(b_1-b_2)^2}{4}.$$

Expanding and simplifying this, we obtain

$$x^2 - (a_1+a_2)x + y^2 - (b_1+b_2)y + a_1a_2 + b_1b_2 = 0. \quad (1)$$

A simpler and more elegant approach can be based on geometric principles. If $Q(x, y)$ is any point on the circle different from P_1 and P_2, then the angle P_1QP_2 is a right angle. Thus, the slopes of P_1Q and P_2Q are negative reciprocals, and we have

$$\frac{y-b_1}{x-a_1} = -\frac{x-a_2}{y-b_2}.$$

This, upon cross-multiplying, immediately leads to the same solution as in (1) above. This method also provides a different geometric insight to this standard problem. This approach can be extended to handle the comparable problem in three dimensions: *find the equation of the sphere, given the endpoints of any diameter.* The standard approach is to parallel the initial argument mentioned above, using analytic geometry. A much simpler approach is to utilize some elementary vector analysis. If $P_1(a_1,b_1,c_1)$ and $P_2(a_2,b_2,c_2)$ are the two given points and $Q(x,y,z)$ is any other point on the surface of the sphere, then the vectors P_1Q and P_2Q must be perpendicular, and so their dot product will be zero. This leads directly to the solution

$$x^2 - (a_1+a_2)x + a_1a_2 + y^2 - (b_1+b_2)y + b_1b_2 + z^2 - (c_1+c_2)z + c_1c_2 = 0.$$

Editor's Note: The same type of argument [see, for example, page 112 of C. H. Lehmann's *Analytic Geometry*, John Wiley & Sons, 1942] can be used to prove that any angle P_1QP_2 inscribed in a circle is a right angle. $P_1(-r,0)$ and $P_2(r,0)$ determine the circle $x^2+y^2=r^2$. For any point $Q(x, y)$ on this circle, P_1Q has slope $m_1 = y/(x+r)$ and P_2Q has slope $m_2 = y/(x-r)$. Hence, $m_1m_2 = y^2/(x-r)^2 = -1$.
