This is easily proved formally by induction, using equation (1). For if we assume that equation (2) holds for some n, then

$$\begin{aligned} p_{n+2} &= \frac{p_{n+1}+p_n}{2} \\ &= \frac{p_{n+1}}{2} + \frac{1}{2}\left(p_{n+1} - \frac{(-1)^n(p_1-p_2)}{2^{n-1}}\right) \\ &= p_{n+1} + \frac{(-1)^{n+1}(p_1-p_2)}{2^n}. \end{aligned}$$

To complete the solution, we iterate equation (2) to get a finite geometric series:

$$\begin{aligned} p_{n+1} &= p_n + \frac{(-1)^n(p_1-p_2)}{2^{n-1}} \\ &= p_{n-1} + \left(\frac{(-1)^{n-1}}{2^{n-1-1}} + \frac{(-1)^n}{2^{n-1}}\right)(p_1-p_2) \\ &\;\;\vdots \\ &= p_1 + \left(\frac{(-1)^1}{2^{1-1}} + \cdots + \frac{(-1)^{n-1}}{2^{n-1-1}} + \frac{(-1)^n}{2^{n-1}}\right)(p_1-p_2) \\ &= p_1 - \left[1 + \cdots + (-1/2)^{n-2} + (-1/2)^{n-1}\right](p_1-p_2) \\ &= p_1 - \left(\frac{1-(-1/2)^n}{1-(-1/2)}\right)(p_1-p_2). \end{aligned}$$

Therefore, as n approaches ∞, p_{n+1} approaches a limiting value of $p_1-(\frac{2}{3})(p_1-p_2) = \frac{1}{3}p_1 + \frac{2}{3}p_2$, or \$1666.676 in our example.

Finally, we note that if Billy asks initial price p_1 and Beth really wants to buy for price p, then her first offer p_2 should be chosen so that $p = \frac{1}{3}p_1 + \frac{2}{3}p_2$. That is, $p_2 = (3p - p_1)/2$. So since Billy asked \$2000, if Beth wanted to pay \$1600 for the car her first offer should have been \$1400.

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A Recurrence Relation in the Spinout Puzzle

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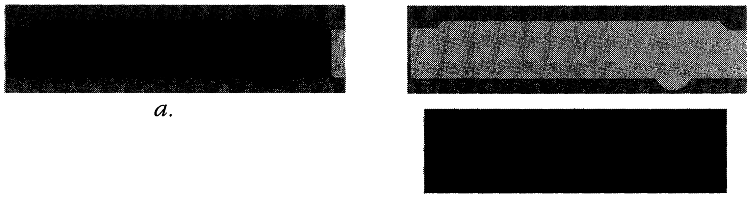
Recently I was introduced to a delightful puzzle from William Keister [**3**] called *spinout,* a modern day version of the Chinese Ring Puzzle. While the Chinese Ring Puzzle has been discussed in several places in recent years [**1**], [**2**], spinout is not as well known. It is a nice way to introduce recursion in discrete mathematics or computer science classes.

The spinout puzzle consists of seven gates attached to a moveable bar, which is encased in a frame. The puzzle starts with all of the gates in the vertical position (Figure 1a), and the objective is to turn all the gates one quarter-turn counterclockwise so that the bar with all gates horizontal can be slid to the right out of the frame (Figure 1b). A gate can be turned only when it is above the semicircular opening at the bottom of the frame. The semicircular opening is positioned on the frame so that there is room for one vertical gate to the right of the gate you want to turn. Also, because of their shapes, you cannot turn a gate if the gate to its right is hori-



a. *b.*

Figure 1. The spinout puzzle.

zontal (Figure 2). These constraints mean that the gates must be turned in a specific sequence.

Although the actual puzzle has seven gates, the key idea in the recursive solution strategy I will describe is to consider the more general puzzle having n gates. Also, an important observation for solving the puzzle is that each move is invertible, so that if a sequence of moves solves the puzzle, the reverse sequence of inverse moves would restore it to its original state.

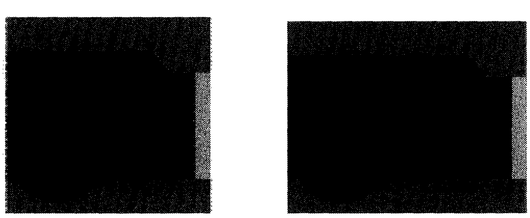
To begin, note that a puzzle with no gates is solved with zero turns, since the bar can just be slid out. The puzzle with one gate is solved simply by turning this gate horizontally and then sliding the bar out. To see how the restrictions apply, notice that if the puzzle has two gates then we must turn the left gate first and the right gate second in order to slide out the bar in two turns. These solutions use the minimal number of gate turns.

Now let F_k denote the minimal number of gate turns possible in a solution of the k-gate puzzle, and assume we have found such a minimal solution for each $k < n$. Then Figure 3 shows the steps to follow to solve the n-gate puzzle using F_n turns, where

$$F_n = F_{n-1} + 2F_{n-2} + 1. \tag{1}$$

Starting from the initial configuration (a), with no fewer than F_{n-2} gate turns we can reach configuration (b) without moving the two gates on the left. It takes one gate turn to go from (b) to (c). Then, again without moving the two gates on the left, we can reverse the first sequence of F_{n-2} moves to transform (c) to (d). But then, leaving the left-most gate fixed, we need only perform no fewer than F_{n-1} gate turns to complete the solution of the puzzle!

This solution of the n-gate puzzle uses the fewest turns possible. If the solution were not the minimal one, then there would have been at least one superfluous turn



a. *b.*

Figure 2. Two interlocking gates.

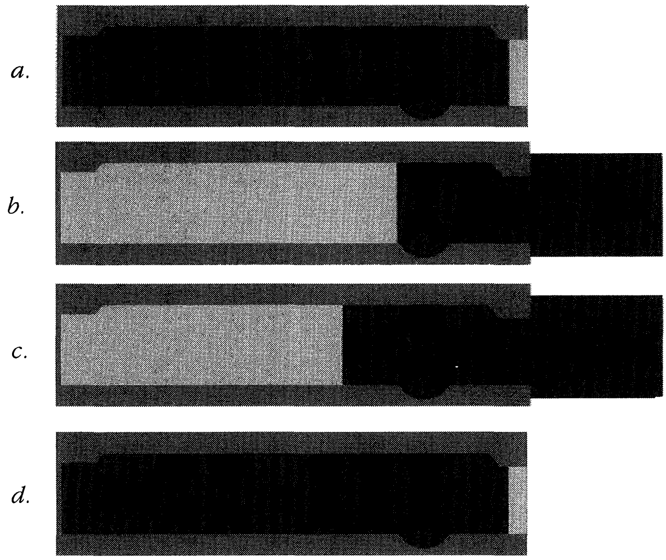


Figure 3. Spinout reduction.

in going from configuration (a) to (b), or from (d) to the solution. But this is contrary to our assumption that we found a minimal solution of the k-gate puzzle for each $k < n$. Hence, by induction, the solution is minimal for all $n \geq 0$.

We could now use our recursive formula, starting with $F_0 = 0$ and $F_1 = 1$, to successively find $F_2, F_3, \ldots, F_7$, but it is not difficult to solve the difference equation (1) to find a closed formula for F_n.

Adding $F_{n-1} + 1$ to both sides of (1), we get

$$F_n + F_{n-1} + 1 = 2(F_{n-1} + F_{n-2} + 1).$$

Iteration yields

$$F_n + F_{n-1} + 1 = 2^n,$$

or

$$F_n + F_{n-1} = 2^n - 1. \tag{2}$$

Next we replace n by $2k$ in equation (1):

$$F_{2k} = F_{2k-1} + 2F_{2k-2} + 1,$$

then subtract F_{2k-2} from both sides and use (2) to obtain

$$\begin{aligned} F_{2k} - F_{2k-2} &= F_{2k-1} + F_{2k-2} + 1 \\ &= 2^{2k-1} - 1 + 1 = 2^{2k-1}. \end{aligned}$$

Since this holds for all k, it follows that

$$\begin{aligned}
F_{2k} - F_0 &= (F_{2k} - F_{2k-2}) + (F_{2k-2} - F_{2k-4} + \cdots + (F_2 - F_0) \\
&= 2^{2k-1} + 2^{2k-3} + \cdots + 2^5 + 2^3 + 2 \\
&= 2(2^{2k-2} + 2^{2k-4} + \cdots + 2^4 + 2^2 + 1) \\
&= 2(4^{k-1} + 4^{k-2} + \cdots + 4^2 + 4 + 1) \\
&= \frac{2}{3}(4^k - 1).
\end{aligned}$$

Therefore, recalling that $F_0 = 0$,

$$F_{2k} = \frac{2}{3}(4^k - 1). \tag{3}$$

We are now going to find a closed expression for F_{2k-1}. Replacing n by $2k$ in equation (2), we get $F_{2k} + F_{2k-1} \pm 2^{2k} - 1$, and so $F_{2k-1} = 2^{2k} - F_{2k} - 1$. By (3), we have

$$F_{2k-1} = 4^k - \frac{2}{3}(4^k - 1) - 1 = \frac{1}{3}(4^k - 1), \tag{4}$$

the desired expression.

Since $4^k = 2^{2k} = 2^n$, formulas (3) and (4) can be expressed in the form

$$F_n = \begin{cases} \frac{2}{3}(2^n - 1), & \text{for } n \text{ even;} \\ \frac{1}{3}(2^{n+1} - 1), & \text{for } n \text{ odd.} \end{cases} \tag{5}$$

But the second component of F_n in (5) may be rewritten as

$$\frac{1}{3}(2^{n+1} - 1) = \frac{2}{3}2^n - \frac{1}{3} = \frac{2}{3}(2^n - 1) + \frac{1}{3}.$$

Therefore, we can replace the right-hand side of (5) by the ceiling function to yield the single formula

$$F_n = \left\lceil \frac{2}{3}(2^n - 1) \right\rceil,$$

valid for all n. In particular, the 7-gate puzzle requires $F_7 = 85$ turns.

For other techniques for solving equation (1), see [**2**].

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