$$3^4 \approx 2^4 \cdot 5 \approx 2^4 \cdot 2^{7/3} = 2^{19/3} \text{ yields } 3 \approx 2^{19/12} = 2^{1.58333\ldots},$$

which is better than $3 \approx ((3+1)(3-1))^{1/2} = 2^{3/2}$.

$$5^2 \approx (5+1)(5-1) = 2^3 \cdot 3 \approx 2^3 \cdot 2^{19/12} = 2^{55/12} \text{ yields } 5 \approx 2^{55/24} = 2^{2.291666\ldots},$$

which is not as good as $5 \approx 2^{7/3}$.

$$7 \cdot 3^2 = 63 \approx 2^6 \text{ yields } 7 \approx 2^6/(2^{19/12})^2 = 2^{17/6} = 2^{2.8333\ldots},$$

which is better than $7 \approx 2^{11/4}$. These examples suggest that there is instructional value in seeking approximate relations that yield better approximations 2^x (x, rational) for the primes.

These techniques also lend themselves to compound interest problems, where equations such as $(1.12)^t = 1.4$ frequently arise. Since

$$1.12 = (7 \cdot 2^2)/5^2 \approx (2^{17/6} \cdot 2^2)/2^{14/3} = 2^{1/6}$$

and

$$1.4 = 7/5 \approx 2^{17/6}/2^{7/3} = 2^{1/2},$$

we find that $2^{t/6} \approx 2^{1/2}$ and $t \approx 3.0$. Since $t = 2.9689944\ldots$, via logs, our result is accurate to the nearest tenth.

A General Method of Deriving the Auxiliary Equation for Cauchy-Euler Equations

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The object of this note is to present a method for obtaining the auxiliary equation associated with the Cauchy-Euler linear differential equation of nth order

$$c_n x^n y^{(n)} + c_{n-1} x^{n-1} y^{(n-1)} + \cdots + c_1 x y^{(1)} + c_0 y^{(0)} = 0 \qquad (x > 0), \quad (\text{CE})_n$$

where c_i $(0 \leqslant i \leqslant n)$ are constants with $c_n = 1$ and where

$$y^{(0)} = y(x) \quad \text{and} \quad y^{(i)} = \frac{d^i}{dx^i}\{y(x)\} \qquad \text{for} \quad i = 1, 2, \ldots, n.$$

For large values of n (specifically, for $n \geqslant 4$), the method described in textbooks to derive the auxiliary equation is time-consuming and laborious. We believe that the following approach is simple and elegant; it enables one to write the general solution of any Cauchy-Euler linear differential equation with considerable ease.

The usual method for deriving the auxiliary equation associated with $(\text{CE})_n$ is to assume that $y(x) = x^m$ is a solution of $(\text{CE})_n$. Then

$$y^{(i)} = m(m-1)(m-2)\cdots(m-i+1)x^{m-i},$$

and substitution of these equations for $y^{(i)}$ into $(\mathrm{CE})_n$ gives us a polynomial equation of nth degree in m. The procedure for obtaining this equation in m (referred to as the auxiliary equation associated with $(\mathrm{CE})_n$) is elementary, but tedious for large values of n.

For a more efficient approach to obtaining the auxiliary equation, we note that the "Factorial Polynomial" $m(m-1)(m-2)\cdots(m-i+1)$ can be written in the form

$$a_{i,1}m + a_{i,2}m^2 + a_{i,3}m^3 + \cdots + a_{i,i}m^i,$$

where the $a_{i,j}$ are called the Stirling Numbers of the First Kind. [See, for example, M. Abramovitz and I. A. Stegun's *Handbook of Mathematical Functions*, p. 833.] Now define the lower triangular matrix (of Stirling numbers)

$$A_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_{2,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & 1 \end{bmatrix},$$

where

$$a_{i,1} = (-1)^{i-1}(i-1)! \quad \text{and} \quad a_{i+1,j} = -ia_{i,j} + a_{i,(j-1)}\,.$$

Observe, for example, that:

$$A_{2\times 2} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \qquad A_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}, \qquad A_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{bmatrix}.$$

Because of the manner in which $A_{n\times n}$ was defined, we can readily write the auxiliary equation associated with $(\mathrm{CE})_n$ as

$$m^n + k_{n-1}m^{n-1} + k_{n-2}m^{n-2} + \cdots + k_1 m + c_0, \qquad (\mathrm{AE})_n$$

where

$$k_r = \sum_{i=r}^{n} a_{i,r}c_i = (c_r, c_{r+1}, \ldots c_n)\begin{bmatrix} a_{r,r} \\ a_{r+1,r} \\ \vdots \\ a_{n,r} \end{bmatrix}$$

for $r = 1, 2, \ldots, (n-1)$. Thus, k_r is the inner product of $(c_r, c_{r+1}, \ldots c_n)$ with the nonzero entries of the rth column of matrix $A_{n\times n}$.

The auxiliary equation for

$$c_2x^2y^{(2)} + c_1xy^{(1)} + c_0y = 0 \qquad (\mathrm{CE})_2$$

(here $c_2 = 1$), obtained via $A_{2\times 2}$, is

$$m^2 + k_1m + c_0 = 0, \qquad (\mathrm{AE})_2$$

where

$$k_1 = (c_1, c_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = c_1 - 1.$$

Hence, $(AE)_2$ is $m^2 + (c_1 - 1)m + c_0 = 0$.

The auxiliary equation for

$$c_3x^3y^{(3)} + c_2x^2y^{(2)} + c_1xy^{(1)} + c_0y = 0 \qquad (CE)_3$$

(here $c_3 = 1$), obtained via $A_{3\times 3}$, is

$$m^3 + k_2m^2 + k_1m + c_0 = 0, \qquad (AE)_3$$

where

$$k_1 = (c_1, c_2, c_3)\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = c_1 - c_2 + 2$$

$$k_2 = (c_2, c_3)\begin{pmatrix} 1 \\ -3 \end{pmatrix} = c_2 - 3.$$

Hence, $(AE)_3$ is $m^3 + (c_2 - 3)m^2 + (c_1 - c_2 + 2)m + c_0 = 0$.

The auxiliary equation

$$m^4 + k_3m^3 + k_2m^2 + k_1m + c_0 = 0 \qquad (AE)_4$$

for

$$c_4x^4y^{(4)} + c_3x^3y^{(3)} + c_2x^2y^{(2)} + c_1xy^{(1)} + c_0y = 0 \qquad (CE)_4$$

(where $c_4 = 1$) can be readily rewritten, using the columns of $A_{4\times 4}$, as

$$m^4 + (c_3 - 6)m^3 + (c_2 - 3c_3 + 11)m^2 + (c_1 - c_2 + 2c_3 - 6)m + c_0 = 0.$$

Example 1. Write the general solution of

$$x^4y^{(4)} - 4x^3y^{(3)} + 12x^2y^{(2)} - 24xy^{(1)} + 24y = 0 \qquad (x > 0).$$

This is $(CE)_4$ with $c_3 = -4$, $c_2 = 12$, $c_1 = -24$, $c_0 = 24$. Therefore, as was clear in rewriting $(AE)_4$,

$$k_3 = c_3 - 6 = -10$$

$$k_2 = c_2 - 3c_3 + 11 = 35$$

$$k_1 = c_1 - c_2 + 2c_3 - 6 = -50.$$

Thus, the auxiliary equation is

$$m^4 - 10m^3 + 35m^2 - 50m + 24 = 0.$$

Since the roots of this equation are $m_1 = 1$, $m_2 = 2$, $m_3 = 3$, and $m_4 = 4$, the general solution of the given Cauchy-Euler equation is

$$y(x) = c_1x + c_2x^2 + c_3x^3 + c_4x^4 \qquad (x > 0).$$

Example 2. Find the general solution of

$$x^5y^{(5)} + 7x^4y^{(4)} + 9x^3y^{(3)} - 6x^2y^{(2)} - 5xy^{(1)} - 3y = 0 \qquad (x > 0).$$

This is $(CE)_5$ with $c_4 = 7$, $c_3 = 9$, $c_2 = -6$, $c_1 = -5$, $c_0 = -3$. Using $A_{5\times5}$, we obtain

$$k_1 = c_1 - c_2 + 2c_3 - 6c_4 + 24 = 1$$
$$k_2 = c_2 - 3c_3 + 11c_4 - 50 = -6$$
$$k_3 = c_3 - 6c_4 + 35 = 2$$
$$k_4 = c_4 - 10 = -3.$$

Therefore, the auxiliary equation is

$$m^5 - 3m^4 + 2m^3 - 6m^2 + m - 3 = 0.$$

This has one real root $m_1 = 3$ and two complex roots $m_2 = -i$, $m_3 = +i$, each with multiplying two. Hence, for $x > 0$:

$$y(x) = c_1x^3 + c_2\cos\ln x + c_3\sin\ln x + \ln x[c_4\cos\ln x + c_5\sin\ln x].$$

Theory versus Reality

If a coin falls heads repeatedly one hundred times, then the statistically ignorant would claim that the 'law of averages' must almost compel it to fall tails next time; any statistician would point out the independence of each trial, and the uncertainty of the next outcome.
But any fool can see that the coin must be double headed.

Ludwik Drazek (1982)

If there is a 50-50 chance that something can go wrong, then 9 times out of 10 it will.

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