Integer-Sided Triangles with One Angle Twice Another

R. S. Luthar, University of Wisconsin Center, Janesville, WI

We offer a simple characterization, in terms of a triangle's sides, of the condition that one angle is twice another. Our derivation is a nice way to combine several key results in trigonometry. As further enrichment for students, we apply our characterization to the case where the sides of a triangle are to be natural numbers. In doing so, we make use of a key result of elementary number theory.

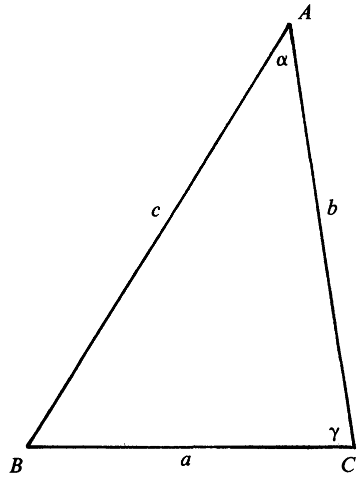

Figure 1.

Suppose γ is twice α. From the *law of sines*, $\dfrac{c}{\sin 2\alpha} = \dfrac{a}{\sin\alpha}$, we obtain

$$c = 2a\cos\alpha. \tag{1}$$

Then, using the *law of cosines*, there follows

$$c = 2a\sqrt{\frac{1+\cos 2\alpha}{2}} = 2a\sqrt{\frac{1+(a^2+b^2-c^2)/2ab}{2}} = a\sqrt{\frac{(a+b)^2-c^2}{ab}}\ .$$

Squaring and simplifying, we obtain

$$ab = c^2 - a^2. \tag{2}$$

Thus, (2) is a necessary condition for γ to be twice α. That it is also sufficient can be seen as follows. From (2) and the *law of cosines*, we have

$$\cos\gamma = \frac{b^2+a^2-c^2}{2ab} = \frac{b^2-ab}{2ab} = \frac{b-a}{2a}$$

and

$$\cos\alpha = \frac{b^2+c^2-a^2}{2bc} = \frac{b^2+ab}{2bc} = \frac{b+a}{2c} = \frac{ab+a^2}{2ac} = \frac{c^2}{2ac} = \frac{c}{2a}\ .$$

Therefore, $\gamma = 2\alpha$ follows from

$$\cos 2\alpha = 2\cos^2\alpha - 1 = \frac{c^2}{2a^2} - 1 = \frac{b-a}{2a} = \cos\gamma,$$

and so $\gamma = 2\alpha$.

To obtain positive integral solutions of (2), we recast it in the more familiar form

$$x^2 + y^2 = z^2 \qquad (y, \text{ even}) \tag{3}$$

and use the fact (see any standard text on number theory) that (3) has solutions

$$x = m^2 - n^2, \qquad y = 2mn, \qquad z = m^2 + n^2, \tag{4}$$

where $m > n > 0$. Taking $r = a$ and $s = (a + b)$ in the identity

$$rs = \left(\frac{r+s}{2}\right)^2 - \left(\frac{r-s}{2}\right)^2,$$

and rearranging terms, we can write (2) as

$$b^2 + (2c)^2 = (2a + b)^2.$$

Therefore, from (4), we have

$$b = m^2 - n^2, \qquad c = mn, \qquad a = n^2.$$

The table below hints at a pattern of $n - 1$ triangles that exist for each $n > 1$.

m	n	a	b	c
3	2	4	5	6
4	3	9	7	12
5	3	9	16	15
⋮	⋮	⋮	⋮	⋮
$n+1$	n	n^2	$(n+1)^2 - n^2$	$n(n+1)$
$n+2$	n	n^2	$(n+2)^2 - n^2$	$n(n+2)$
⋮	⋮	⋮		
$2n-1$	n	n^2	$(2n-1)^2 - n^2$	$n(2n-1)$

To verify that there are exactly $n - 1$ triangles for each integer $n > 1$, observe that

$$a + c > b \Leftrightarrow n^2 + mn > m^2 - n^2 \Leftrightarrow 2n > m.$$

Since $m > n$, the inequality $2n > m$ also requires that $n > 1$.