$\int \sin\theta \sin^n(m\theta)\,d\theta$ and $\int \cos\theta \sin^n(m\theta)\,d\theta$ which Grant tackles using integration by parts. (Incidentally, checking the example above and a few others by differentiation may prompt some to notice the forms that appear as antiderivatives and thereby to sense the possibility of yet another method: undetermined coefficients.)

A Circular Argument

Fred Richman, Florida Atlantic University, Boca Raton, FL 33431-0991

Sketch of the circle. The first interesting limit that the student of calculus is exposed to is often

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \qquad (*)$$

This limit has received some attention recently, see [**5**], [**6**], [**9**] and [**13**]. It is not usually recognized that the standard proof is circular, as was suggested in [**13**] and denied in [**9**]. Archimedes proved a variant of $(*)$ in order to show that the area of a circle is equal to the area of a right triangle whose perpendicular sides are the radius and the circumference of the circle. By the definition of π, the circumference of a circle is $2\pi r$, so his theorem establishes the area formula πr^2. Despite this dependence of the area formula on $(*)$, the area formula is the basis for the argument used in most calculus texts to prove $(*)$; see [**3**], [**7**], [**10**], [**11**], [**12**].

The usual proof. The standard argument for $(*)$ hinges on the inequalities

$$\sin x < x < \tan x, \qquad (**)$$

from which $(*)$ readily follows. The usual proof of $(**)$ considers an arc AB of length x on the circle of radius one with center at O, and the point B' on the extension of OB such that AB' is perpendicular to OA. The triangle OAB is contained in the circular sector OAB which is contained in the triangle OAB'. Moreover

(1) the area of the triangle OAB is $(\sin x)/2$,
(2) the area of the circular sector OAB is $x/2$,
(3) the area of the triangle OAB' is $(\tan x)/2$.

Statements (1) and (3) are clearly true; Statement (2) is true because the area of the sector is to the area π of the circle as the length x of the arc AB is to the circumference 2π of the circle.

What's wrong with this proof? The problem lies in how we know that the area of the circle is π. The answer that we learned it in elementary school is not good enough. The fact is that to prove that the area of the circle is π, we have to invoke $(*)$ in some form; for example, in the form of the inequalities $(**)$.

Archimedes' proof that the area of a circle is πr^2. Archimedes was perhaps the first to prove that the area of a circle of radius r is πr^2. Euclid had shown earlier [**4**; XII.2] that the area of a circle is *proportional to* r^2. Archimedes inscribes and circumscribes the circle with regular n-sided polygons. The length of a side of the inscribed polygon is, in our terms, $2\sin \pi/n$, the length of a side of the circumscribed polygon is $2\tan \pi/n$, and $2\pi/n$ is the length of the circular arc between adjacent points of contact of the circle with either polygon.

Archimedes argues that $2\sin\pi/n < 2\pi/n$ because the shortest distance between two points is a straight line. This principle is intuitively clear, despite the fact that Euclid felt it necessary to prove that any side of a triangle was shorter than the sum of the other two [**4**, I.20]. To prove that $2\pi/n < 2\tan\pi/n$ Archimedes invokes the following more complicated principle [**2**, p. 145]:

($***$) If two plane curves C and D with the same endpoints are concave in the same direction, and C is included between D and the straight line joining the endpoints, then the length of C is less than the length D.

This is not an implausible principle—I find it rather attractive. Still, it doesn't have the immediacy of "the shortest distance between two points is a straight line," and I am not sure that it is easier to accept than ($*$) itself.

Principle ($***$) is applied with C the circular arc between two adjacent points of contact of the circumscribed polygon with the circle, and D the polygonal path consisting of the two adjacent halves of the sides of the circumscribed polygon that touch the circle at these points. Thus *Archimedes proves* ($**$) *in order to show that the area of a circle is* πr^2. He needs ($**$) to show that the lengths of the perimeters of the polygons approximate the length of the circumference of the circle.

Related proofs. In the proof of ($*$) in [**8**, Chap. 10, Sec. 2], Johnson and Kiokemeister tacitly assume that if A and B are points on a circle, P is a point outside the circle, and PA and PB are tangent to the circle, then the length of the arc from A to B is less than the sum of the lengths of PA and PB. So their proof is like that of Archimedes—and is not circular—but they do not justify this step.

In [**9**] Rose shows that ($*$) is equivalent to the validity of the area formula $sr/2$ for a circular sector of radius r and arc length s. Nevertheless, he claims that the standard proof of ($*$) "need not involve circular reasoning since the sector area formula can be obtained geometrically." Indeed Archimedes obtained the formula geometrically, but he first had to establish ($**$); the standard proof uses the area formula to show ($*$), completing the circle. The alternative is to assume ($*$) by *defining* arc length to be the supremum of the lengths of inscribed polygonal paths.

True by definition? Ultimately the problem may be that ($*$) is true by definition, as suggested by Gillman in [**6**]: "the theorem on the limit $(\sin\alpha)/\alpha$ in its natural setting as essentially just the definition of the circumference of a circle." The idea underlying the definition of arc length is that small chords approximate small arcs, so polygonal paths can be used to approximate curves. There is no explanatory value in proving ($*$) because ($*$) is presupposed in the definition of arc length. The best we can do is provide an informal motivation for ($*$). Drawing a few isosceles triangles OAA' on a fixed base AA', with the angle at O equal to $2x$ getting smaller and smaller, might convince a student that the length $2rx$ of the circular arc AA', with center at O, approaches the length $2r\sin x$ of the line AA'. Getting across an intuitive feeling for the idea that small arcs are approximately linear is worthwhile in any case: that idea, after all, is the basis for the notion of a derivative.

An ingenious solution. Apostol [**1**, page 102] circumvents circularity by using area rather than arc length to define the radian measure of an angle. He does not mention the problem of circularity explicitly; instead he comments that he has on hand a general notion of area via the definite integral, but not yet a general notion of arc length. The measure of an angle is defined to be twice the area of the circular sector it subtends, divided by the square of the radius. Then $(**)$ follows immediately from the inclusion of the areas because $x/2$ is *defined* to be the area of the sector. Later, when arc length is defined as the limit of the lengths of polygonal approximations, it can be shown that this definition of the measure of an angle agrees with the standard one that uses arc length.

Although Apostol's solution is elegant, the use of a nonstandard definition of angle measure is a serious drawback in a calculus course. Nevertheless, a case can be made that plane area is a more accessible concept than arc length, and so provides a better way to measure angles. Area is readily bounded by polygons from above and below; arc length, absent a principle like $(***)$, is bounded by polygonal lines only from below. If a circle can be put between two polygons, then we can be confident that the area of the circle—whatever it might be—lies between the areas of those two polygons. But why couldn't the circumference of a circle be greater than the supremum of the perimeters of its inscribed polygons?

Recap. The usual proof of $(*)$ uses the area formula to prove $(**)$. The classical proof of the area formula uses $(**)$, which is established by appeal to the unfamiliar, and not quite obvious principle $(***)$. The modern alternative to postulating $(***)$ is to define arc length as the supremum of the lengths of polygonal approximations, but this amounts to postulating $(*)$.

There are various ways out.

- Continue being circular. After all, a proof is just a completely convincing argument. The students accept the area formula, so why not use it?
- Use Archimedes' $(***)$, explicitly or tacitly. Maybe it's obvious—Johnson and Kiokemeister evidently thought that it was.
- Postulate a suitable form of $(*)$. In the form of the definition of arc length, this is the current view of a logical development. Moreover Gillman says that deriving the area formula from this "is what so pleased the students."
- Define angle measure using area, as Apostol does, thus postponing the whole question of arc length.

References

1. T. M. Apostol, *Calculus*, Blaisdell, Waltham, Mass., 1967.
2. E. J. Dijksterhuis, *Archimedes*, Princeton University Press, Princeton, NJ, 1987.
3. C. H. Edwards, and D. E. Penney, *Calculus and Analytic Geometry*, third edition, Prentice Hall, Englewood Cliffs, NJ, 1990.
4. Euclid, *The Elements*, Translated by T. L. Heath, Dover, New York, 1956.
5. W. B. Gearhart, and H. S. Shultz, The function sin x/x, *CMJ* 21 (1990) 90–99.
6. L. Gillman, π and the limit of $(\sin\alpha)/\alpha$, *American Mathematical Monthly* 98 (1991) 345–348.
7. R. A. Hunt, *Calculus with Analytic Geometry*, Harper & Row, New York, 1988.
8. R. E. Johnson, and F. L. Kiokemeister, *Calculus with Analytic Geometry*, third edition, Allyn and Bacon, 1969.
9. D. A. Rose, The differentiability of sin x, *CMJ* 22 (1991) 139–142.
10. S. K. Stein, *Calculus and Analytic Geometry*, fourth edition, McGraw-Hill, New York, 1987.
11. E. W. Swokowski, *Calculus*, fifth edition, PWS-KENT, Boston, 1991.
12. G. B. Thomas, and R. L. Finney, *Calculus and Analytic Geometry*, sixth edition, Addison-Wesley, Reading, MA 1984.
13. P. Ungar, Reviews, *American Mathematical Monthly* 93 (1986) 221–230.