Combining the four cases, we obtain the probability distribution for this random variable X:

$$P(X=n)=$$
$$\sum_{i=2}^{n-9}\sum_{j=1}^{n-i-8}\sum_{k=6}^{n-2-i-j}\frac{(n-1)!}{1!\,i!\,j!\,k!(n-2-i-j-k)!}\left(\frac{1}{6}\right)^{2+i+j+k}\left(\frac{2}{6}\right)^{n-2-i-j-k}$$
$$+\sum_{i=2}^{n-9}\sum_{j=2}^{n-i-7}\sum_{k=6}^{n-1-i-j}\frac{(n-1)!}{i!\,j!\,k!(n-1-i-j-k)!}\left(\frac{1}{6}\right)^{1+i+j+k}\left(\frac{2}{6}\right)^{n-1-i-j-k}$$
$$+\sum_{i=2}^{n-9}\sum_{j=1}^{n-i-8}\sum_{k=6}^{n-2-i-j}\frac{(n-1)!}{1!\,i!\,j!\,k!(n-2-i-j-k)!}\left(\frac{1}{6}\right)^{2+i+j+k}\left(\frac{2}{6}\right)^{n-2-i-j-k}$$
$$+\sum_{i=2}^{n-9}\sum_{j=2}^{n-i-7}\sum_{k=1}^{n-6-i-j}\frac{(n-1)!}{5!\,i!\,j!\,k!(n-6-i-j-k)!}\left(\frac{1}{6}\right)^{6+i+j+k}\left(\frac{2}{6}\right)^{n-6-i-j-k} \quad (7)$$

for $n=11,12,\ldots$.

Recall that the expected number of tosses required to earn the body and head is 12. Now we want to add to this the expected value of the probability distribution X represented by (7). Thus, the expected number of tosses to build the entire cootie is

$$12+E(X)=12+\sum_{n=11}^{\infty}nP(X=n).$$

By tailoring a *Mathematica* program, we were able to calculate this value and found it to be 48.953478 (with eight significant digits), consistent with the value obtained in the simulations.

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Minimal Pyramids

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Here is a natural optimization problem that might be given to the better students of a first-year calculus course. Some characteristics of its solution are mildly surprising.

Find the dimensions of the pyramid of minimum volume whose base is a regular n-gon and whose base and triangular faces are all tangent to a fixed sphere.

Figure 1 shows a slice of the pyramid through the apex and perpendicular to an edge of the base polygon. The radius of the inscribed sphere is r, the height of the pyramid is h, and a is the apothem—the distance from the center to the midpoint of a side of the base.

We wish to express the volume of the pyramid in terms of the constant r and the variable h.

Using similar triangles, we have

$$\frac{a}{h}=\frac{r}{\sqrt{h(h-2r)}}$$

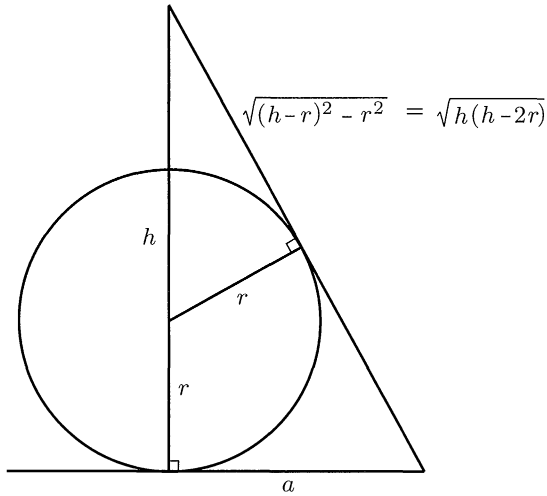

Figure 1

so

$$a = r\sqrt{\frac{h}{h-2r}}. \tag{1}$$

To express the area of the base of the pyramid in terms of h and r, let b be half the length of a side of the base n-gon and let $2\beta = 2\pi/n$ be the central angle subtended by a side of the base, as in Figure 2.

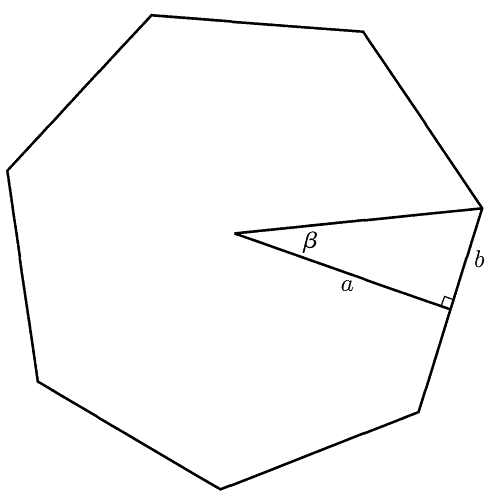

Figure 2

Since $\tan\beta = b/a$ and $\beta = \pi/n$, the area A of the base is $2n$ times the area of the triangle in Figure 2. That is,

$$A = 2n\left(\frac{1}{2}a^2\tan\left(\frac{\pi}{n}\right)\right) = na^2\tan\left(\frac{\pi}{n}\right). \tag{2}$$

Substituting (1) into (2), we arrive at

$$A = \frac{nr^2h}{h-2r}\tan\left(\frac{\pi}{n}\right).$$

The volume of any cone is given by $\frac{1}{3}Ah$, where h is its height and A is the area of its base. Thus, the volume V of our pyramid is given by

$$V = \frac{1}{3}Ah = \frac{nr^2h^2}{3(h-2r)}\tan\left(\frac{\pi}{n}\right).$$

Note that the domain of the function $V(h)$ is the open interval $2r < h < \infty$, and the volume increases without bound as h approaches either endpoint of this interval. To minimize V we set the derivative

$$\frac{dV}{dh} = \frac{nr^2}{3}\tan\left(\frac{\pi}{n}\right)\frac{h(h-4r)}{(h-2r)^2}$$

equal to zero and find that the minimum volume is attained when $h = 4r$. Somewhat surprisingly, our answer is independent of the number n of sides in the base! Since the minimizing condition $h = 4r$ holds for any value of n, then by letting n go to infinity it follows that the circular cone of minimum volume circumscribed about a sphere of radius r will also have a height equal to $4r$.

For an additional exercise, find the dimensions of the pyramid of minimum surface area whose base is a regular n-gon and whose base and triangular faces are all tangent to a fixed sphere.

Taylor Polynomials for Rational Functions

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How would you calculate the third-order Taylor polynomial for

$$f(x) = \frac{x^4+x^2+2}{x^3+x+1}$$

at the origin? In the usual treatments of Taylor polynomials and Taylor's theorem, rational functions $f(x) = P(x)/Q(x)$ for polynomials P and Q are largely ignored. I imagine this is due to the difficulty of calculating higher-order derivatives of these functions. Here is a way to calculate Taylor polynomials for rational functions that is simple computationally and conceptually. Moreover, as a byproduct we will explicitly calculate the remainders and show that they have a useful form. We can use these explicitly calculated polynomials and associated remainders as concrete examples and motivation for Taylor's theorem.

To find the third-order Taylor polynomial for the above example at the origin, we must first check that the denominator does not vanish at the origin. It does not, so we can proceed with simple long division. For long division of polynomials, we usually arrange the terms of both polynomials in order of decreasing degree, as in the following computation: