contradiction. It is not hard to extend this proof to arbitrary roots of arbitrary positive integers.

If n and k are integers greater than one, and n is not the kth power of an integer, then $n^{1/k}$ is irrational.

Suppose $n^{1/k}$ is rational. Then $n^{i/k}$ is rational for each positive integer i, and so there exists a sequence of positive integers a_i such that each $a_i n^{i/k}$ is an integer. Then $a = a_1 a_2 \cdots a_{k-1}$ is a positive integer, and $an^{i/k}$ is integral for $1 \leqslant i \leqslant k-1$. Let b be the smallest positive integer such that $bn^{i/k}$ is an integer for $1 \leqslant i \leqslant k-1$. Since n is not a kth power, there is an integer m such that $m < n^{1/k} < m+1$. Let $c = bn^{1/k} - bm$. Then c is a positive integer less than b. But for $1 \leqslant i \leqslant k-2$,

$$cn^{i/k} = bn^{(i+1)/k} - bn^{i/k}m$$

is an integer. And for $i = k-1$,

$$cn^{(k-1)/k} = bn - bn^{(k-1)/k}m$$

is an integer. This contradicts the definition of b, and completes the proof.

The proofs usually given for this result make some use of the Fundamental Theorem of Arithmetic, which may require too much time to explain (let alone to prove) in, say, a precalculus course. As Professor Niven observed, the fact that this proof assumes the well-ordering principle has never been seen to bother a student.

The Derivatives of Arcsec x, Arctan x, and Tan x

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Beginning calculus students know that the shaded sector in Figure 1 has area equal to $(1/2)r^2\theta$. Here we show how different expressions for θ can yield formulas for the above titled derivatives.

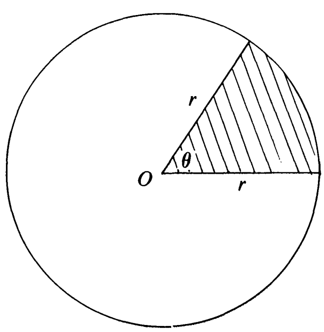

Figure 1.

To obtain the formula for $d(\text{arcsec}\, x)/dx$, let (Figure 2) BE and CD be arcs of circles with center O and radii x and $x + \Delta x$, respectively. Then,

$$\text{area(sector } OBE) < \text{area(triangle } OBD) < \text{area(sector } OCD). \qquad (*)$$

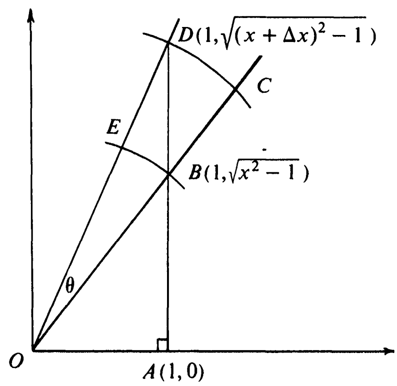

Figure 2.

Since

$$\theta = \operatorname{arcsec}(x + \Delta x) - \operatorname{arcsec} x,$$

it follows from (*) that

$$x^2\left[\operatorname{arcsec}(x + \Delta x) - \operatorname{arcsec} x\right] < \sqrt{(x + \Delta x)^2 - 1} - \sqrt{x^2 - 1}$$

$$< x + \Delta x)^2[\operatorname{arcsec}(x + \Delta x) - \operatorname{arcsec} x].$$

Thus,

$$\frac{\sqrt{(x + \Delta x)^2 - 1} - \sqrt{x^2 - 1}}{(\Delta x)(x + \Delta x)^2} < \frac{\operatorname{arcsec}(x + \Delta x) - \operatorname{arcsec} x}{\Delta x}$$

$$< \frac{\sqrt{(x + \Delta x)^2 - 1} - \sqrt{x^2 - 1}}{(\Delta x)(x^2)},$$

or

$$\frac{2x + \Delta x}{(x + \Delta x)^2\left[\sqrt{(x + \Delta x)^2 - 1} + \sqrt{x^2 - 1}\right]} < \frac{\operatorname{arcsec}(x + \Delta x) - \operatorname{arcsec} x}{\Delta x}$$

$$< \frac{2x + \Delta x}{x^2\left[\sqrt{(x + \Delta x)^2 - 1} + \sqrt{x^2 - 1}\right]}.$$

Consequently,

$$\frac{d(\operatorname{arcsec} x)}{dx} = \lim_{\Delta x \to 0} \frac{\operatorname{arcsec}(x + \Delta x) - \operatorname{arcsec} x}{\Delta x} = \frac{1}{x\sqrt{x^2 - 1}}. \tag{1}$$

Using (1), it is easy to obtain the derivatives of the other inverse trigonometric functions. For example, $d(\arctan x)/dx = d(\operatorname{arcsec}\sqrt{x^2 + 1})/dx = 1/(1 + x^2)$.

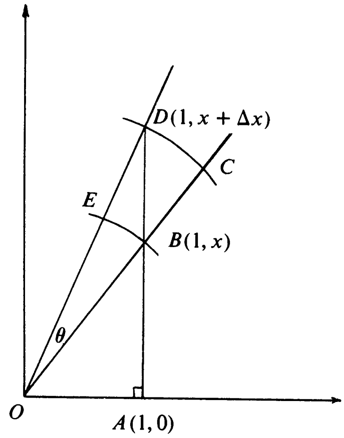

Figure 3a.

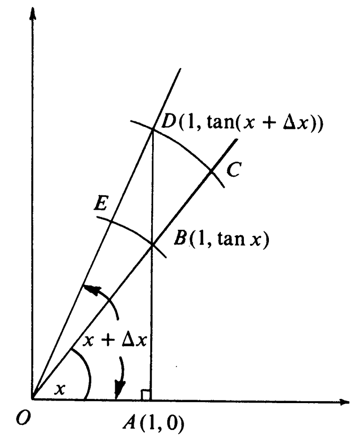

Figure 3b.

To obtain the formula for $d(\arctan x)/dx$, let radius $OB = \sqrt{1 + x^2}$ and radius $OC = \sqrt{1 + (x + \Delta x)^2}$ in Figure 3a. Then

$$\theta = \arctan(x + \Delta x) - \arctan x$$

and (*) yields

$$(1 + x^2)[\arctan(x + \Delta x) - \arctan x] < \Delta x < [1 + (x + \Delta x)^2] \cdot [\arctan(x + \Delta x) - \arctan x].$$

Therefore,

$$\frac{1}{1 + (x + \Delta x)^2} < \frac{\arctan(x + \Delta x) - \arctan x}{\Delta x} < \frac{1}{1 + x^2},$$

and we have

$$\frac{d(\arctan x)}{dx} = \lim_{\Delta x \to 0} \frac{\arctan(x + \Delta x) - \arctan x}{\Delta x} = \frac{1}{1 + x^2}. \qquad (2)$$

Now it is a simple matter to obtain the derivatives of the trigonometric functions by way of formulas (1) or (2). For instance, take $y = \tan x$ so that $x = \arctan y$. Then $dx/dy = 1/(1 + y^2)$ and $dy/dx = 1 + y^2 = \sec^2 x$.

To obtain the formula for $d(\tan x)/dx$, let radius $OB = \sec x$ and radius $OC = \sec(x + \Delta x)$ in Figure 3b. Then $\theta = \Delta x$ and (*) yields

$$(\sec^2 x)(\Delta x) < \tan(x + \Delta x) - \tan x < [\sec^2(x + \Delta x)](\Delta x).$$

Thus,

$$\frac{d(\tan x)}{dx} = \lim_{\Delta x \to 0} \frac{\tan(x + \Delta x) - \tan x}{\Delta x} = \sec^2 x.$$