the ellipse in S onto the picture plane P is an ellipse. Thus, the question addressed by this paper is answered in the affirmative.

What's Significant about a Digit?

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The question of the title might well have been (but wasn't) posed by a student wondering about instructions to report numerical answers to, say, "four significant digits." (At Duke, we routinely give such instructions to our calculus students.) It will come as no surprise that our freshmen enter with little or no idea as to what makes a digit significant.

In spite of the fact that practically every standard calculus book assumes that students will use calculators at least some of the time, you will look in vain for one that defines significant digit. Small wonder then that students make quite arbitrary decisions, reporting everything displayed on a calculator as a final answer, but discarding digits to suit their convenience if an intermediate result must be copied down and rekeyed.

What then are significant digits? A reliable source [*The American Heritage Dictionary of the English Language*, *New College Edition* (W. Morris, ed.), Houghton Mifflin, 1978] defines them as:

> The digits of the decimal form of a number beginning with the leftmost nonzero digit and extending to the right to include all digits warranted by the accuracy of measuring devices used to obtain the number.

That definition, while clearly aimed at laboratory science applications, may be the only one our students have seen in secondary school (if, indeed, they have seen any definition at all). It's actually a workable definition for a mathematics course, if we expand "measuring" to "measuring and calculating." It is still an intuitive definition, however, because of the imprecision of "warranted."

The natural place to look for a mathematical definition of significant digit would be in a numerical analysis book—indeed, it is the "creeping down" of topics from numerical analysis into the calculus sequence that makes this topic important at this time. But not even the authors of these books agree on the "correct" definition. On page 5 of Anthony Ralston's classic text [*A First Course in Numerical Analysis*, McGraw-Hill, 1965], we find:

> If y is any approximation to a true value x, then the k-th decimal place of y is said to be *significant* if
>
> $$|x-y| < 0.5 \times 10^{-k}.$$
>
> Therefore, every digit of a correctly rounded number is significant.

This definition works equally well for digits to the left or right of the decimal point if we don't require k to be positive; that is, if we allow the units place to correspond to $k=0$, the tens place to $k=-1$, and so on. However, there is no mention of "leading zeros," and the author follows the definition with a paragraph to explain how this leads to difficulty in determining when numbers are "equally significant." The paragraph ends, "We shall therefore avoid the use of the notion of the number of significant digits in a number."

R. L. Burden and J. D. Faires' more recent, and currently popular, text [*Numerical Analysis* 3rd ed., Prindle, Weber, and Schmidt, 1985] eschews the "intuitive" definition and substitutes one based on *relative* error. Except for notation (changed to match that in Ralston's text), we find on page 12:

> A number y approximates a number x to t *significant digits* if t is the largest nonnegative integer such that
>
> $$\frac{|x-y|}{|x|} < 5 \times 10^{-t}.$$

One might question whether the multiplying factor should be "5" or "0.5," but the examples make it clear that "5" is what the authors intend. For example, the definition implies that y gives a 4 SD approximation to $x = 1000$ if $999.5 < y < 1000.5$. But wait! According to this definition, y gives a 4 SD approximation to $x = 999$ if $998.5005 < y < 999.4995$. The presumed virtue in calling the latter example 4 SD rather than 3 SD is that their concept of SD is "continuous" in the sense that for fixed t, the endpoints of the interval in which y approximates x to t SD vary continuously with x. However, the price paid to achieve continuity, in violation of intuition, seems too high. Essentially the same definition is given by S. D. Conte and C. de Boor [*Elementary Numerical Analysis*: *An Algorithmic Approach*, 3rd ed., McGraw-Hill, 1980, p. 10], but the authors immediately contradict it with their examples. For instance, they state that 3 is a 1 SD approximation to π, whereas the definition asserts that it is a 2 SD approximation.

The formal definition that underlies the usual intuitive approach to significant digits is the following:

(a) Leading zeros are *never* significant.

(b) The kth decimal digit (reading from left to right) of an approximation y to a number x is *significant* if it is not a leading zero, and

$$|x-y| < 0.5 \times 10^{-k}.$$

(c) The approximation y to x has t *significant digits* if the first t of its nonzero digits are significant in the sense defined in (b).

See, for example, G. Dahlquist and A. Bjorck's *Numerical Methods*, Prentice Hall, 1974, p. 24.

The following examples will illustrate the application of this definition to the use of a calculator.

Example 1. An eight-place calculator might display $\pi/40$ as 0.0785398, or as .07853982, or as $7.8539816 \times 10^{-02}$, depending on the way it uses its eight digits. But (as can be seen from a more accurate solution), the first display has 6 SD, the second has 7 SD, and the third has 8 SD, even though eight digits are being shown in all cases. The 4 SD answer is 0.07854. It requires *five* decimal places, and it can be obtained from any of the three displays shown.

Example 2. An eight digit display of π^{10} is 93648.047. The 4 SD approximation is 93650. Trailing zeros are sometimes necessary even when they are not significant.

Example 3. An eight digit answer for 20013/10006 is 2.0000999. The 4 SD answer is 2.000, *not* 2, or even 2.0. Sometimes trailing zeros are significant, and they must be present to indicate the level of accuracy after rounding. (By the way, appearances notwithstanding, 2.0001 is *not* the exact answer from this division; the next digit after the three 9's is 4.)

Example 4. Suppose you want a decimal approximation to $\pi/\sqrt{2}$. The best answer available from an eight-place calculator is 2.2214415, and the 4 SD answer is 2.221. Now suppose you had used 1.41 to approximate $\sqrt{2}$ and 3.14 to approximate π. The calculator would then show the result of the division as 2.2269504. This answer contains 8 SD of the rational fraction 314/141, but it does not have even *three* correct SD's of the number you were looking for. Rounding to four places gives 2.227, which is off by 6 in the fourth digit; rounding to three places gives 2.23, whereas the correct 3 SD answer is 2.22. *Moral*: In general, you cannot expect your answer to contain more SD's than your least accurate input, and it may contain fewer. That is why you should *not* discard "extra" digits in an intermediate result. Use your calculator's memory as much as possible. If you must copy an intermediate result and key it in again, copy and rekey all of it.

Example 5. When subtracting numbers that are close together, you can lose SD's very quickly. Suppose you subtract π from 355/113. The fraction is displayed as 3.1415929, and π is displayed as 3.1415927. When you subtract these numbers, your display may show .00000027, or 2.667×10^{-7}, or even 1.8×10^{-7}. (These rather different looking answers all came from *actual* eight-place calculators.) How many digits are significant in each of these answers?

The first seven SD's of the two numbers we subtracted matched exactly; that is, they cancelled in the subtraction. If we had done the calculation on paper, our answer would have been .0000002, which couldn't possibly have more than one significant digit. As it turns out, even that one is wrong. Where did all those other digits come from? If we had a 12-digit calculator, this is the way subtraction would have appeared:

$$3.14159292035 - 3.14159265359 = .00000026676.$$

Thus, the first answer given had 2 SD, the second had 3 SD (not four), and the third had none (even after rounding). We emphasize that all three answers came from calculators that show eight digits; the differences were in "hidden digits," and if you don't know how many of those your calculator has, you can't count on them.

When the Duke Mathematics Department first discussed making "4 SD" the standard requirement for numerical answers, there were objections along the following lines: An answer such as $8(10\sqrt{10} - 1)/27$ is exact. Why should we settle for 9.073, which is only an approximation? The symbolic answer makes it easy to tell whether the student got the right answer. An incorrect symbolic answer also can reveal what the student did incorrectly.

Any response to the above objections should begin by noting that the symbolic answer is both *exact* and essentially *meaningless*. The decimal answer is meaningful to the student (as an order of magnitude for the quantity being computed) and even potentially useful (as a computational step in solving an engineering or physics problem). The decimal answer can also be related to estimation techniques for checking the plausibility of the answer—something we ought to be teaching, but often do not.

Symbolic answers may seem easier to check because we have been doing it for a long time. We overlook the fact that there are infinitely many correct symbolic forms for a given answer. (Any large group of test-takers will discover at least a few that hadn't occurred to us.) There is essentially *only one* correct 4 SD answer—the infinity of variations beyond that number of places don't have to be checked! The choice of "4" is arbitrary, but it is large enough to eliminate guessing, and it is small enough to override differences among calculators or (in most cases) disastrous cancellations.

Of course, we usually cannot tell how a student arrived at a wrong decimal approximation. (Since it is in the nature of calculus problems that the student's resort to the calculator comes late in the computation, an incorrect symbolic result is also probably close at hand.) However, it is possible that a somewhat less accurate decimal approximation may result from greater understanding of mathematics than that being tested. For example, a student required to calculate a definite integral, but who cannot remember the appropriate antiderivative, might resort to the trapezoidal rule, with enough steps to get 2 or 3 SD. This approach should be rewarded, not penalized, since it demonstrates a better understanding of integration than most of our students ever acquire.

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Finding Rational Roots of Polynomials

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In "Synthetic Division Shortened" [*TYCMJ* 12 (November 1981) 334–336], Warren Page and Leo Chosid gave a very useful necessary condition for a polynomial with integral coefficients to have a rational root. In this capsule, we provide two additional results designed to ease the work involved in finding rational roots of polynomials with integer coefficients. Although both of these results are known, neither seems to be readily available in the literature. The proofs given here are quite simple.

Let us begin by stating the rational root theorem.

Theorem 1. *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial all of whose coefficients are integers. If $f(p/q) = 0$ for relatively prime integers p and q, then $p|a_0$ and $q|a_n$.*

The procedure for finding the rational roots of the polynomial $f(x)$ is to list all possible rational numbers p/q such that $p|a_0$ and $q|a_n$, and to see which, if any, satisfy $f(p/q) = 0$. Of course, this task isn't quite as arduous as it looks, since we can use Descartes' rule of signs and results on upper and lower bounds for the zeros to eliminate the need to check every possibility. And those rationals that need to be tested can be checked rather quickly by the Page-Chosid method alluded to above. However, if a_0 and a_n have many factors, there could still be many rational numbers to check.

Our first result says that if certain conditions are fulfilled, then the polynomial has no rational roots.

Theorem 2. *Let $f(x)$ be a polynomial of degree at least two defined as in Theorem 1. If a_0, a_n and $f(1)$ are all odd, then $f(x)$ has no rational roots.*