An Alternative for Certain Partial Fractions

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In the January 1983 Classroom Capsules column, J. E. Nymann showed that the indefinite integral

$$\int \frac{\alpha x^n + \beta}{x(\gamma x^n + \delta)} dx \tag{1}$$

could be evaluated without using partial fractions. His method involved multiplying the numerator and denominator by x^m and then determining m so that the resulting integrand was in the form $\frac{du}{ku}$. Since neither the formula for m nor that for k is "nice," one might feel that a still better method ought to exist! We offer another, simpler route to evaluating (1). Our approach also relates integrals of this form to a standard exercise in differential equations courses, thereby illustrating a second path to the answer.

The simple route to evaluating (1) is to first separate it into two integrals

$$\alpha \int \frac{x^n}{x(\gamma x^n + \delta)} dx + \beta \int \frac{x^{-n}}{x(\gamma + \delta x^{-n})} dx. \tag{2}$$

If one substitutes $z_1 = x^n$ into the first of these integrals, and $z_2 = x^{-n}$ into the second, each reduces to an integral in the form

$$\int \frac{dz}{az + b},$$

and therefore can be integrated without further difficulty. These substitutions are not only easy to remember and apply, they also obviate the need for partial fractions on certain integrals which are not amenable to Nymann's strategy. For example, either $z = x^n$ or $z = x^{-n}$ would reduce

$$\int \frac{x^n}{x(\gamma x^{2n} + \delta)} dx$$

to integration forms that students should recognize.

As for the connection with differential equations, suppose we rewrite the second integral in (2) as

$$y(x) = \int \frac{\beta}{x(\gamma x^n + \delta)} dx. \tag{3}$$

Then (by differentiating with respect to x, taking reciprocals, and rearranging) one finds that the inverse function $x = x(y)$ satisfies

$$\frac{dx}{dy} - \frac{\delta}{\beta} x = \frac{\gamma}{\beta} x^{n+1}. \tag{4}$$

Students in beginning differential equations courses should recognize (4) as a standard Bernoulli-type first-order differential equation. Some of them might enjoy solving it for $x = x(y)$ using techniques from their differential equations course, and then comparing that answer with its inverse $y = y(x)$ obtained by integrating (3) via the substitution $z = x^{-n}$.

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Deriving the Equations of the Ellipse and Hyperbola

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Bill Bompart [TYCMJ 13 (1982) 198] describes a method for solving radical equations by using appropriate substitutions to transform the radical equation to a system of equations. The purpose of this note is to apply the same technique to the derivation of the equations of the ellipse and hyperbola.

Let $F_1(-c,0)$ and $F_2(c,0)$ be the foci of the ellipse centered at the origin, and let $P(x,y)$ be any point on the ellipse. Then by the definition of an ellipse

$$d_1 + d_2 = 2a, \tag{1}$$

where $d_1 = F_1P$ and $d_2 = F_2P$. Using the distance formula,

$$d_1^2 = (x+c)^2 + y^2 \quad \text{and} \quad d_2^2 = (x-c)^2 + y^2, \tag{2}$$

we obtain

$$d_1^2 - d_2^2 = 4cx. \tag{3}$$

From (1) and (3), it follows that

$$d_1 - d_2 = \frac{2cx}{a}. \tag{4}$$

Solving (1) and (4), we obtain

$$d_1 = a + \frac{c}{a}x \quad \text{and} \quad d_2 = a - \frac{c}{a}x \tag{5}$$

which are important in their own right since they give rational expressions for the distances from any point of the ellipse to its foci. Combining (2) and (5), we have

$$(x+c)^2 + y^2 = \left(a + \frac{c}{a}x\right)^2 \quad \text{and} \quad (x-c)^2 + y^2 = \left(a - \frac{c}{a}x\right)^2.$$

To obtain the equation of the ellipse, just simplify either equation after substituting b^2 for $a^2 - c^2$. In a similar manner, the equation for the hyperbola can be derived.

Equations (5) provide some additional information. Observing that $c/a = \epsilon$, where ϵ is the eccentricity of an ellipse, we write (5) as

$$d_1 = a + \epsilon x \quad \text{and} \quad d_2 = a - \epsilon x. \tag{6}$$