more (or less) likely [see L. S. Sgroi and R. J. Sgroi, *Mathematics for Elementary School Teachers: Problem-solving Investigations*, PWS-Kent, Boston, 1993]. In higher grades, a more formal analysis like mine may be appropriate, perhaps even generalizing to a greater number of strings.

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Unifying a Family of Extrema Problems

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One of our goals as mathematics teachers is to help students see how to bring unity to problems that seem at first glance to be rather different. In this article we show how Lagrange multipliers can be used to solve a general problem that unifies several common Calculus I and Calculus III extrema problems. The solutions to these extrema problems share an interesting feature which we think is not widely recognized. In addition, we exhibit an elegant proof of the arithmetic–geometric means inequality.

Fence problem. We begin with a Calculus I problem that illustrates some of the features of the more general problem. We seek to minimize the total cost of fencing to enclose a rectangular area of 5184 square meters. The cost of fencing varies by sides as follows: \$4.00 per meter and \$5.00 per meter along the two north-south boundaries and \$1.50 per meter and \$2.50 per meter along the two east-west boundaries.

This problem is easily solved by standard methods; we will not repeat the solution here. However, a graphical analysis is instructive.

Let x be the length in the north-south direction and $C(x)$ be the total cost. Also, let $S(x)$ be the cost of the north-south fencing, and $W(x)$ be the cost of the east-west fencing. Then $S(x) = 9x$, $W(x) = 4(5184/x)$, and $C(x) = S(x) + W(x)$. As Figure 1 illustrates, $C(x)$ is a minimum when $S(x) = W(x)$. In other words, the

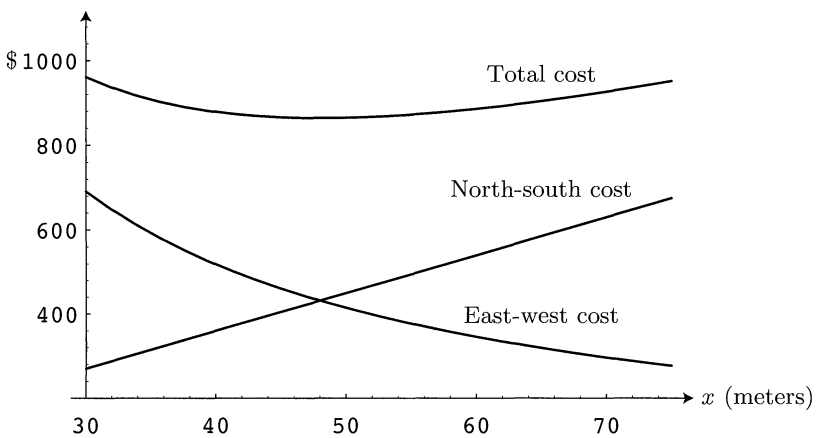

Figure 1

total cost of fencing is a minimum when the cost of east-west fencing is equal to the cost of north-south fencing. Variants of this remarkable result appear in each of the following problems.

In the fence problem we minimize $u + v$ subject to the constraint that uv is a constant. We have shown graphically that the minimum of $u+v$ occurs when $u = v$. The dual problem, which is to maximize uv subject to the constraint that $u + v$ is a constant, has the same solution, $u = v$. The use of Lagrange multipliers makes it easy to solve the corresponding problems for any number of variables.

Problem 1. Maximize $x_1x_2 \cdots x_n$ subject to the constraint that $x_1 + x_2 + \cdots + x_n$ is a constant.

Solution. The maximum occurs when $x_1 = x_2 = \cdots = x_n$. In order to obtain the solution with Lagrange multipliers, first set $g(x_1, x_2, \ldots, x_n) = x_1+x_2+\cdots+x_n = B$, where B is a constant, and set $f(x_1, x_2, \ldots, x_n) = x_1x_2 \cdots x_n$.

To find where the maximum occurs, we find λ and $x_1, x_2, \ldots, x_n$ so that $\nabla f = \lambda \nabla g$ and $g(x_1, x_2, \ldots, x_n) = B$. That amounts to solving $\lambda x_k = x_1x_2 \cdots x_n$ for $k = 1, 2, \ldots, n$ and $x_1 + x_2 + \cdots + x_n = B$. It is easy to see that the solution is $x_1 = x_2 = \cdots = x_n = B/n$ and $\lambda = (B/n)^{n-1}$. Hence, $x_1x_2 \cdots x_n$ attains its maximum when $x_1 = x_2 = \cdots = x_n$. □

The inequality of the arithmetic and geometric means follows easily from problem 1. Let $A = (x_1 + x_2 + \cdots + x_n)/n$ and $G = \sqrt[n]{x_1x_2 \cdots x_n}$. To prove that $G \leq A$ we note first that $x_1 + x_2 + \cdots + x_n = nA$. So the maximum for $x_1x_2 \cdots x_n$ occurs when $x_1 = x_2 = \cdots = x_n = (nA)/n = A$. Hence $x_1x_2 \cdots x_n \leq A^n$, and $G \leq A$ follows by taking nth roots of both sides.

The dual of problem 1 has the same solution. We state it as problem 2 for reference.

Problem 2. Minimize $x_1 + x_2 + \cdots + x_n$ subject to the constraint that $x_1x_2 \cdots x_n$ is a constant.

Solution. The minimum occurs when $x_1 = x_2 = \cdots = x_n$. □

Before considering some additional standard problems from calculus, we state a useful corollary of problem 2.

Problem 3. Let $a_1, a_2, \ldots, a_n$ be constants. Minimize $a_1/x_1 + a_2/x_2 + \cdots + a_n/x_n$ subject to the constraint that $x_1x_2 \cdots x_n$ is a constant.

Solution. The minimum occurs when $a_1/x_1 = a_2/x_2 = \cdots = a_n/x_n$. To see this, let $z_k = a_k/x_k$ for $k = 1, 2, \ldots, n$. Note that $z_1z_2 \cdots z_n$ is now constant because $x_1x_2 \cdots x_n$ is constant. Hence, by problem 2, the minimum of $z_1 + z_2 + \cdots + z_n$ occurs when $z_1 = z_2 = \cdots = z_n$. □

Box problem. Minimize the cost of a box, with different cost per unit area for each side, subject to the constraint that the volume of the box is a constant.

Solution. The total cost will be at minimum when the costs of each of the three pairs of opposite sides are all equal. This follows from problem 3. Let V be a constant and $V = xyz$, with x, y, and z the dimensions of the box. Note that ayz, bxz, and cxy are just the costs of the three opposite pairs of walls for appropriate constants

a, b, and c. Then $ayz + bxz + cxy$ is the total cost. Since $V = xyz$, the total cost is $aV/x + bV/y + cV/z$. Hence, we minimize $aV/x + bV/y + cV/z$ with $xyz = V$. □

The related problems of a box, with or without a top, having fixed volume and minimal surface area are just special cases of the box problem. Also, many of the usual problems to find the minimal length of fencing needed for a rectangular fixed area of land are special cases of the fence problem solved above.

Of course, the dual problem of maximizing the volume of a box with a fixed total cost and given costs per unit of surface area for the sides has the same solution. That is, if F is the fixed total cost, then the volume is maximized if and only if each pair of opposite sides has a cost of $F/3$. Similarly, the dual problem of finding the maximal rectangular area given a fixed cost of fencing has a symmetric solution.

We now consider a problem and solution suggested by the editor, to whom we are grateful for many useful suggestions.

Cylinder problem. Assume the cost per unit area is given for the top, bottom, and side of a cylinder. Minimize the cost of the cylinder subject to the constraint that the volume is a constant.

Solution. The total cost will be minimized when the cost of the side equals twice the costs of the top and bottom together. This also follows from problem 3. Let b be the cost per unit area for the top and bottom and s be the cost per unit area of the side. Consider minimizing $b\pi r_1 r_2 + s\pi r_1 h + s\pi r_2 h$ under the constraint that $r_1 r_2 h$ is a constant. Here we are thinking of h as the height while r_1 and r_2 are initially thought of as independent variables, which will become the radius after the constraint $r_1 = r_2$ is applied. Proceed as in the box problem to show that the minimum occurs when $b\pi r_1 r_2 = s\pi r_1 h = s\pi r_2 h$. Now apply the additional, but compatible, constraint $r_1 = r_2$ and we have the solution. □

Note that while the solution to the cylinder problem is not exactly analogous to the solution of the box problem as we stated it above, it is true in both cases that the minimum cost occurs when the vertical sides cost twice as much as the top and bottom together.

Other problems. Solutions to other standard problems exhibit the same unifying principle. For instance, if heat loss from the walls, including the roof and floor, of a rectangular building is considered to be proportional to the area of the wall times a "loss constant" (which might vary for each wall), the shape of the most efficient building is such that the heat losses from each opposite pair of walls are equal. Or, consider an example from economics in which a fixed amount of money is used to purchase an amount A of apples and an amount B of bananas, each with its own cost. The utility (the total benefit) is sometimes considered to be proportional to $(AB)^r$, and it is maximized when the amount of money spent on A equals the amount spent on B.

We have shown how three general problems can be solved easily using Lagrange multipliers. These general problems provide simple solutions to certain constrained max-min problems. In addition, the solutions to these problems exhibit a remarkable pattern. Finding this pattern can provide the basis of an extended problem or a project for students in Calculus I or III. For instance, Calculus I students might investigate the fence cost problem as a project in which they plot their solution as in Figure 1, formulate a conjecture, and then demonstrate its validity.

Problems 1–3 follow from theorems proved in [**1**], using the inequality of the arithmetic and geometric means. Simmons [**2**] includes a proof of the arithmetic–geometric means inequality as an exercise in his section on Lagrange multipliers.

References

1. Ivan Niven, *Maxima and Minima Without Calculus* (Dolciani Mathematical Expositions), Mathematical Association of America, Washington, DC, 1981.
2. George F. Simmons, *Calculus with Analytic Geometry*, 2nd ed., McGraw-Hill, New York, 1996.

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A Note on the Ratio of Arc Length to Chordal Length

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Suppose f is a differentiable function on $[a, b]$, and consider two points P and Q on the graph of $y = f(x)$. Let $L(\mathrm{P}, \mathrm{Q})$ denote the arc length of the portion of the graph between P and Q, and let $D(\mathrm{P}, \mathrm{Q})$ denote the Euclidean distance between P and Q. *Calculus*, the popular textbook by D. Hughes-Hallett et al. [Wiley, New York, 1994], gives an informal justification that the derivative of the sine function is the cosine function, based on the assumption that the ratio $L(\mathrm{P}, \mathrm{Q})/D(\mathrm{P}, \mathrm{Q})$ tends to unity as Q tends toward P along a circular arc.

Is this assumption generally true? As we shall see, it holds if f has a continuous derivative. However, the ratio need not tend to unity without this or some other restriction on f. I applaud the success of these authors in exposing the ideas of calculus—a success due in part to their willingness to forego unhelpful rigor. But I think instructors should be aware of this gap in the exposition.

Theorem. *If f has a continuous derivative on $[a, b]$, and if* P *and* Q *are points on the graph of $y = f(x)$, then*

$$\lim_{\mathrm{Q}\to\mathrm{P}} \frac{L(\mathrm{P}, \mathrm{Q})}{D(\mathrm{P}, \mathrm{Q})} = 1.$$

Proof. Without loss of generality, we may assume that $[a, b] = [0, 1]$, $f(0) = 0$, and P is the origin. We then have

$$\begin{aligned}
\lim_{\mathrm{Q}\to\mathrm{P}} \frac{L(\mathrm{P}, \mathrm{Q})}{D(\mathrm{P}, \mathrm{Q})} &= \lim_{x\to 0} \frac{\int_0^x \sqrt{1 + y'^2}\, dt}{\sqrt{x^2 + y^2}} \\
&= \lim_{x\to 0} \frac{\sqrt{1 + y'^2}}{\frac{1}{2}(x^2 + y^2)^{-1/2}(2x + 2yy')} \qquad \text{(l'Hôpital's rule)} \\
&= \lim_{x\to 0} \frac{\sqrt{(1 + y'^2)\,(x^2 + y^2)}}{x + yy'}
\end{aligned}$$