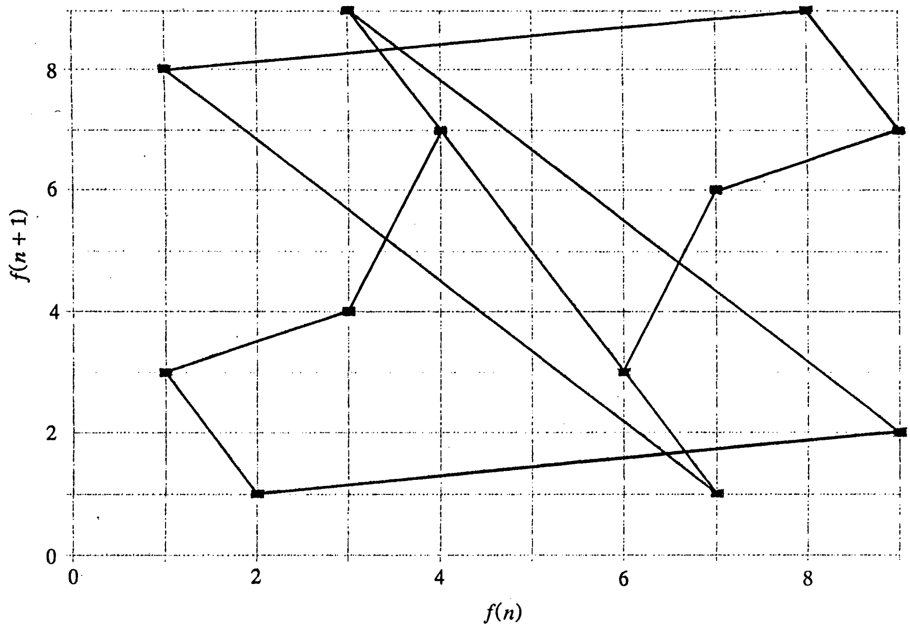


**Figure 2.** The Fibonacci walk for the sequence $1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, \ldots$.

The many questions to consider about the cycle lengths and orbit structures are a fertile source of projects requiring students to formulate and test conjectures.

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## Cubic Splines from Simpson's Rule

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Suppose at the points $x_0, x_1, \ldots, x_n$ we are given data values $y_0, y_1, \ldots, y_n$ and slopes $s_0, s_1, \ldots, s_n$. Then a *cubic Hermite interpolant* is a $C^1$ piecewise cubic curve $y = C(x)$ that interpolates these data values and slopes. In other words, on the data interval $[x_i, x_{i+1}]$ $C(x)$ is the unique cubic polynomial such that $C(x_i) = y_i$, $C'(x_i) = s_i$, $C(x_{i+1}) = y_{i+1}$, and $C'(x_{i+1}) = s_{i+1}$. It is easy to write down an explicit formula for $C(x)$ in each interval. Now suppose the slopes $s_0, s_1, \ldots, s_n$ are not given but are allowed to be chosen arbitrarily. It is a surprising fact that there is a choice of $s_0, s_1, \ldots, s_n$ that produces a cubic Hermite interpolant that is also $C^2$. Such an interpolant is called a *cubic spline*. It is shown in standard textbooks in numerical analysis that, for this to happen, $s_0, s_1, \ldots, s_n$ must satisfy the tridiagonal linear system

$$\begin{bmatrix} 1 & 0 & & & & & \\ h_1 & 2(h_0+h_1) & h_0 & & & & \\ & h_2 & 2(h_1+h_2) & h_1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & h_{n-1} & 2(h_{n-2}+h_{n-1}) & h_{n-2} \\ & & & & 0 & 1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{n-1} \\ s_n \end{bmatrix}$$

$$
= 3 \begin{bmatrix} d_0 \\ \frac{h_1}{h_0}(y_1 - y_0) + \frac{h_0}{h_1}(y_2 - y_1) \\ \frac{h_2}{h_1}(y_2 - y_1) + \frac{h_1}{h_2}(y_3 - y_2) \\ \vdots \\ \frac{h_{n-1}}{h_{n-2}}(y_{n-1} - y_{n-2}) + \frac{h_{n-2}}{h_{n-1}}(y_n - y_{n-1}) \\ d_n \end{bmatrix} \tag{1}
$$

where $h_i = x_{i+1} - x_i$. (To make the problem completely determined, we need conditions at the boundary points. In (1) we have simply assigned the slopes $s_0$ and $s_n$ to have the values $d_0$ and $d_n$. This gives us what is called a *complete cubic spline*. Other boundary conditions are possible, giving rise to other kinds of splines.) The usual derivation of (1) is a tedious algebraic exercise. It is the point of this note to show that there is an easy way to get it.

When we present this material in class, for simplicity we usually restrict the discussion to the equally spaced case. Then (1) becomes

$$
\begin{bmatrix} 1 & & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \cdot & \cdot & \cdot & \\ & & & 1 & 4 & 1 \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{n-1} \\ s_n \end{bmatrix} = \frac{3}{h} \begin{bmatrix} d_0 \\ y_2 - y_0 \\ y_3 - y_1 \\ \vdots \\ y_n - y_{n-2} \\ d_n \end{bmatrix} \tag{2}
$$

where $h = h_0 = h_1 = \cdots = h_{n-1}$. On more than one occasion students have noticed that the coefficients 1, 4, 1 that appear in the matrix also appear in Simpson's rule, and they have naturally asked if there is a connection. It turns out that there is!

Consider the cubic spline $C(x)$ on the double interval $[x_0, x_2]$. Then the fundamental theorem of calculus says $\int_{x_0}^{x_2} C'(x)\,dx = C(x_2) - C(x_0) = y_2 - y_0$. If we could use Simpson's rule to compute the integral on the left exactly, we would obtain

$$
\left( \frac{s_0 + 4s_1 + s_2}{6} \right) 2h = y_2 - y_0, \tag{3}
$$

which is the second equation in (2). All the other equations would follow in the same way. Simpson's rule is exact for cubics, but is it exact for the piecewise quadratic $C'(x)$? A simple geometric argument shows that it is. Let $Q(x)$ be the quadratic polynomial that interpolates the data $(x_0, s_0), (x_1, s_1), (x_2, s_2)$. Then, since $y = C(x)$ is assumed to be $C^2, C'(x) - Q(x)$ is a $C^1$ piecewise quadratic polynomial that interpolates the equally spaced data $(x_0, 0), (x_1, 0), (x_2, 0)$. It is easy to show that its graph must look as in Figure 1, that is, $C'(x) - Q(x)$ must be antisymmetric around $x_1$. We therefore have $\int_{x_0}^{x_2} [C'(x) - Q(x)]\,dx = 0$ or

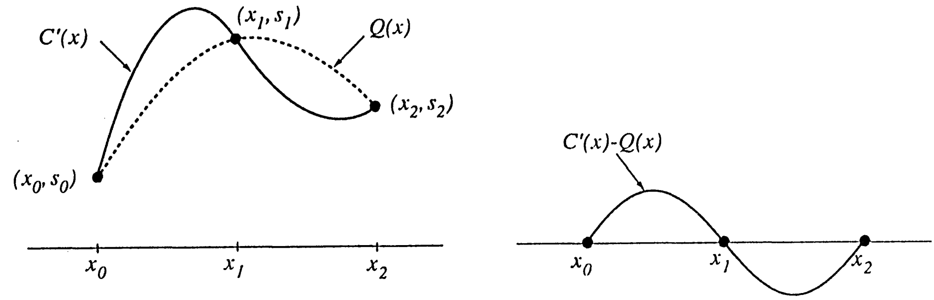


**Figure 1**

$\int_{x_0}^{x_2} C'(x)\,dx = \int_{x_0}^{x_2} Q(x)\,dx$. Since Simpson's rule is exact for the quadratic $Q(x)$, we obtain equation (3) and are done.

The argument above shows that if $C(x)$ is a cubic spline then its slopes must satisfy (2). But for practical purposes, what we really want is the converse of this result. The following argument uses the same idea as above but requires some algebra. We will treat the general case since it does not present any additional difficulties. For simplicity, shift $x_1$ to the origin so that the interval $[x_0, x_2]$ becomes $[-h, k]$. Since $C'(x)$ is piecewise quadratic and passes through the points $(-h, s_0), (0, s_1), (k, s_2)$, then almost by inspection we have

$$C'(x) = \begin{cases} \left(\dfrac{s_0 + \sigma h - s_1}{h^2}\right)x^2 + \sigma x + s_1 & \text{for } x \in [-h, 0] \\ \left(\dfrac{s_2 - \sigma' k - s_1}{k^2}\right)x^2 + \sigma' x + s_1 & \text{for } x \in [0, k] \end{cases}$$

where $\sigma$ and $\sigma'$ are the derivatives of $C'(x)$ at 0 from the left and from the right. Our goal is to find what conditions insure that $\sigma = \sigma'$. If we integrate $C'(x)$ over the intervals $[-h, 0]$ and $[0, k]$ separately, we get the equations

$$\left(\frac{s_0 + \sigma h - s_1}{h^2}\right)\frac{h^3}{3} - \sigma\frac{h^2}{2} + s_1 h = y_1 - y_0$$

$$\left(\frac{s_2 - \sigma' k - s_1}{k^2}\right)\frac{k^3}{3} + \sigma'\frac{k^2}{2} + s_1 k = y_2 - y_1.$$

Multiplying the first equation by $k/h$ and the second by $h/k$ and adding, we get

$$\frac{ks_0 + 2(h+k)s_1 + hs_2}{3} - \frac{hk}{6}(\sigma - \sigma') = \frac{k}{h}(y_1 - y_0) + \frac{h}{k}(y_2 - y_1)$$

which implies that $\sigma = \sigma'$ if and only if

$$ks_0 + 2(h+k)s_1 + hs_2 = 3\frac{k}{h}(y_1 - y_0) + 3\frac{h}{k}(y_2 - y_1).$$

This is exactly the second equation in (1). In this way, we see that (1) implies that a cubic spline $C(x)$ exists.

So the observation that the coefficients in (2) are the same as the coefficients in Simpson's rule has inspired the idea of integrating the derivative of $C(x)$, resulting in a very simple proof that the existence of a cubic spline is equivalent to (1).