converging to $\ln 4$, we would choose only *three* positive terms before the first negative term.

Editor's Note: For related discussions of this theme, see "Rearranging the Alternating Harmonic Series" by C. C. Cowen, K. R. Davidson, and R. P. Kaufman [Amer. Math. Monthly, 87 (December 1980) 817–819], "Sum-Preserving Rearrangements of Infinite Series" by Paul Schaefer [Amer. Math. Monthly, 88 (January 1981) 33–40], and "Rearranging Terms in Alternating Series" by Richard Beigel [Math. Mag., 54 (November 1981) 244–246].

Tetrahedra, Skew Lines, and Volume

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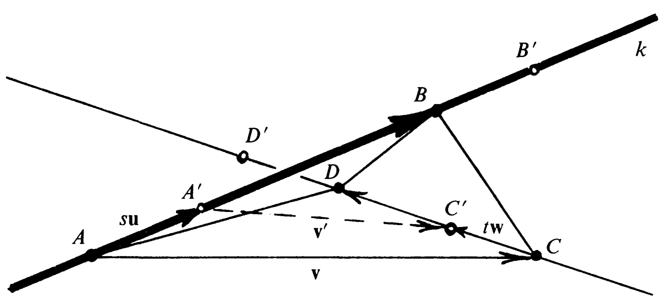
If A and B are distinct points on a line k, and C and D are distinct points on a line m that is skew to k, then A, B, C, and D determine a tetrahedron. This situation can be visualized by cutting a quadrilateral out of a sheet of paper and creasing it along a line between an appropriate pair of vertices. One of the two skew lines is associated with the crease and its two vertices, the other with the two remaining vertices.

Suppose, as above, that points, A, B, C, and D determine a tetrahedron for given fixed skew lines k and m. We intend to show that the volume of the tetrahedron does not change when segments AB and CD of fixed lengths are moved along their respective lines k and m.



To be specific, consider any points A', B' on k and points C', D' on m such that the lengths of segments $A'B'$ and $C'D'$ equal those of segments AB and CD, respectively. We consider the vector $\mathbf{u}'$ from A' to B' as being the same as the vector $\mathbf{u}$ from A to B, and the vector $\mathbf{w}'$ from C' to D' is considered the same vector as $\mathbf{w}$ from C to D.

Since A' lies on line k, the vector $\mathbf{AA}'$ from A to A' satisfies

$$\mathbf{AA}' = s\mathbf{u} \text{ from some scalar } s.$$

Similarly, C' lies on m and so the vector $\mathbf{CC'}$ from C to C' satisfies

$$\mathbf{CC'} = t\mathbf{w} \text{ for some scalar } t.$$

If $\mathbf{v}$ is the vector from A to C, then the vector $\mathbf{v}'$ from A' to C' is given by

$$\mathbf{v}' = \mathbf{v} - s\mathbf{u} + t\mathbf{w}.$$

This is seen by observing that $\mathbf{A'C'} - \mathbf{CC'} = \mathbf{A'C} = \mathbf{AC} - \mathbf{AA'}$, or $\mathbf{v}' - t\mathbf{w} = \mathbf{v} - s\mathbf{u}$.

Let us now compare the volumes of the tetrahedron $ABCD$ and $A'B'C'D'$. The area of triangle ADC is $(1/2)|\mathbf{v} \times \mathbf{w}|$. And since $\mathbf{v} \times \mathbf{w}$ is perpendicular to the plane of triangle ADC, the distance from point B to the plane of triangle ADC is $(1/|\mathbf{v} \times \mathbf{w}|)|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. The volume of tetrahedron $ABCD$, considered as a pyramid with base ADC, is therefore:

$$V = (1/3)\left\{((1/2)|\mathbf{v} \times \mathbf{w}|)\left(\frac{1}{|\mathbf{v} \times \mathbf{w}|}|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|\right)\right\} = (1/6)|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

Using $\mathbf{u}' = \mathbf{u}$ and $\mathbf{w}' = \mathbf{w}$, we similarly obtain the volume of $A'B'C'D'$ as

$$\begin{aligned} V' &= (1/6)|\mathbf{u} \cdot (\mathbf{v}' \times \mathbf{w})| \\ &= (1/6)|\mathbf{u} \cdot [(\mathbf{v} - s\mathbf{u} + t\mathbf{w}) \times \mathbf{w}]| \\ &= (1/6)|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w} - s\mathbf{u} \times \mathbf{w})| \qquad \text{(since } \mathbf{w} \times \mathbf{w} \text{ is the zero vector)} \\ &= (1/6)|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \qquad \text{(since } \mathbf{u} \text{ and } s\mathbf{u} \times \mathbf{w} \text{ are perpendicular, } \mathbf{u} \cdot (s\mathbf{u} \times \mathbf{w}) = 0). \end{aligned}$$

Thus, the tetrahedra have equal volumes.

Another way of thinking through a proof is as follows: Fix AB. As CD moves, the distance from point A to line m does not change. Since neither the height of triangle ADC nor the length of the base (CD) changes, neither does the area. As triangle ADC moves—or is distorted (keeping constant area) in the plane of point A and line m, the distance from point B to that plane does not change. The volume of tetrahedron $ABCD$ is one-third the product of this latter distance and the area of triangle ADC. Moving CD changes neither of these; hence it does not change the volume. Similarly, with DC fixed in a new position, moving AB would not change the original volume.

Example. Suppose k is the z-axis and m is the x-axis in the x-y-z coordinate system, and let

$$\begin{array}{llll} A = (0,0,4) & B = (0,0,5) & C = (3,0,0) & D = (5,0,0) \\ A' = (0,0,-1) & B' = (0,0,0) & C' = (-4,0,0) & D' = (-2,0,0). \end{array}$$

Then $\mathbf{u} = B - A = (0,0,1)$, $\mathbf{w} = D - C = (2,0,0)$, $\mathbf{v} = C - A = (3,0,-4)$, and $\mathbf{v}' = C' - A' = (-4,0,1)$. Thus, the vector equation $\mathbf{v}' = \mathbf{v} - s\mathbf{u} + t\mathbf{w}$ holds for $t = -7/2$ and $s = -5$, even though the lines k and m are not skew.

The skewness property is required for computing the distance from B to the plane of triangle ADC. If the lines k and m were not skew, there would be a plane containing them and hence a plane containing the vectors, $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Since $\mathbf{v} \times \mathbf{w}$ is perpendicular to this plane, and therefore to $\mathbf{u}$, the height of the tetrahedron would be $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|/|\mathbf{v}\times\mathbf{w}| = 0$. Thus, skewness of the lines forces the existence of tetrahedra; but it is not necessary for the validity of the vector equation $\mathbf{v}' = \mathbf{v} - s\mathbf{u} + t\mathbf{w}$.

Proclus Diadochus 412–485

It is well known that the man who first made public the theory of irrationals perished in a shipwreck in order that the inexpressible and unimaginable should ever remain veiled. And so the guilty man, who fortuitously touched on and revealed this aspect of living things, was taken to the place where he began and there is for ever beaten by the waves.

[Attributed] Scholium to Book X of *Euclid* V. 417