Recognizing that the limit as h goes to 0 of the term in the denominator is precisely the definition of the derivative of e^x at $x=0$, we immediately conclude that

$$\int_a^b e^x\,dx = e^b - e^a.$$

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On Laplace's Extension of the Buffon Needle Problem

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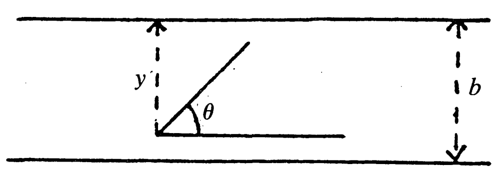
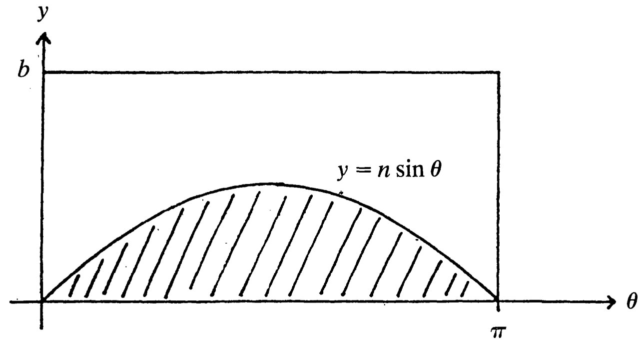
The classical Buffon needle problem is to find the probability that a needle of length n when dropped on a floor made of boards of width b will cross a crack between the boards. This problem can be solved by evaluating a simple single integral. In his extension of the problem, Laplace considered a floor tiled by congruent rectangles and considered the probability of the needle crossing one or two of the cracks between the rectangles. The problem of computing this probability is given as an exercise in some references [**2**, **4**, **13**] the answer is merely stated in others [**1**, **12**], and is discussed in some detail by Solomon [**11**]. The only reference we know of that provides an elementary presentation of the solution of Laplace's problem is in error [**10**], and it is the purpose of this note to provide a complete and correct solution. In our solution, the universe of possible positions in which the needle can fall is modeled on a three-dimensional coordinate system and the problem is solved by some straightforward computations of double integrals. We conclude with a brief survey of the many variations on the Buffon and Laplace needle problems and we suggest some variants to pursue.

The original Buffon needle problem. In the diagram below a needle of length n is depicted on a floor with boards of width b. The distance from the base of the needle to the floorboard "above" the needle is denoted by y. The angle made by the needle with the horizontal is denoted by θ. If we assume that $n < b$, then the universe of positions in which the needle can fall is seen on the following graph with the shaded portion representing the positions where the needle crosses a crack. (The case where $n > b$ is left for the reader.)



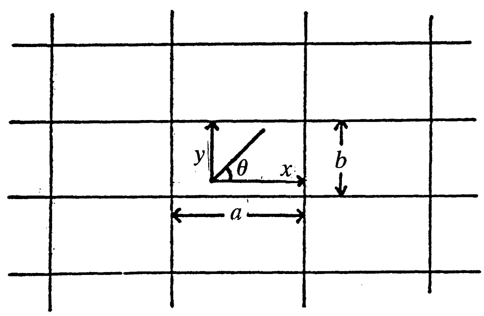
$0 \le y < b$
$0 \le \theta < \pi$



Assuming a uniform probability distribution, the probability of the needle crossing a crack is given by:

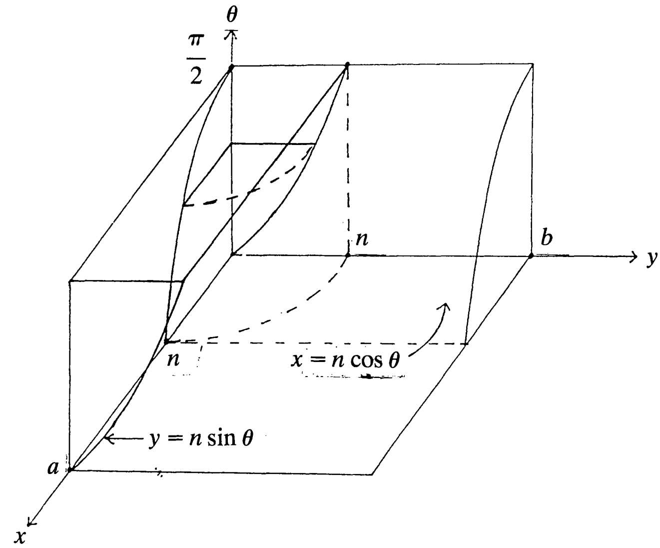
$$P = \frac{\int_0^{\pi/2} n \sin\theta \, d\theta}{b\pi/2} = \frac{2n}{b\pi}.$$

Laplace's extension of the Buffon needle problem. Consider the following diagram in which the needle is shown in a random position on a floor tiled by rectangles of side lengths a and b. The distances from the base of the needle to the next horizontal and vertical cracks are represented by y and x respectively, and θ is the angle the needle makes with the horizontal.



$0 \le x < a$
$0 \le y < b$
$0 \le \theta < \pi$

We assume that n is smaller than both a and b and we leave the other cases to the reader. The possible positions of the needle for which $\theta < \pi/2$ are shown on the following graph in 3-space. If $x < n\cos\theta$ so that $\theta < \arccos(x/n)$, we get a vertical crossing. Thus the region enclosed by the surface $\theta = \arccos(x/n)$ and the XY-plane for $0 \le y < b$ represents those positions in which the needle crosses a vertical crack. Similarly, the region bounded by $\theta = \arcsin(y/n)$ and the ΘX-plane for $0 \le x < a$ represents positions in which the needle crosses a horizontal crack. The intersection of these two regions then represents positions in which the needle



makes both a horizontal and vertical crossing. Once again we mean by randomness that θ, x, and y have uniformly distributed probabilities within their respective ranges. The probabilities of each of these three events can then be obtained by evaluating the following expressions.

For a horizontal crossing,

$$P_h = \frac{\int_0^a \int_0^{\pi/2} n \sin\theta \, d\theta \, dx}{ab\pi/2} = \frac{2n}{b\pi}.$$

By symmetry, for a vertical crossing, we obtain $P_v = 2n/\pi a$.

To compute the probability of crossing both horizontally and vertically we first observe that the intersection curve of our surfaces projects down on a circle, that is, $\arcsin(y/n) = \arccos(x/n)$ implies $y^2 = n^2 - x^2$. Thus

$$P_{h \text{ and } v} = \frac{\int_0^{\pi/2} \int_0^{n\cos\theta} \int_0^{n\sin\theta} dy \, dx \, d\theta}{ab\pi/2}$$

$$= \frac{\int_0^{\pi/2} n^2 \sin\theta\cos\theta \, d\theta}{ab\pi/2} = \frac{n^2/2}{ab\pi/2} = \frac{n^2}{\pi ab}.$$

The probability of crossing either horizontally or vertically can now be computed as the sum of the probabilities of each event minus the probability of both:

$$P_{h \text{ or } v} = \frac{2n}{a\pi} + \frac{2n}{b\pi} - \frac{n^2}{ab\pi} = \frac{2n(a+b) - n^2}{\pi ab}.$$

Note that as $a \to \infty$, we obtain the solution to the original Buffon needle problem.

The probability of the needle falling entirely within one of the rectangles is then

$$1 - P_{h\ or\ v} = \frac{\pi ab - 2n(a+b) + n^2}{\pi ab}$$

One can simulate the Buffon needle experiment with a computer, and there are many variations on the Buffon or Laplace needle problems that can be pursued as calculus exercises or with the assistance of a computer. For example, instead of parallel lines, one might try a collection of n lines through a single point with uniform angular spacing between the lines [**3**]. One might consider other tilings of the plane (see [**6**, **9**]), say by hexagons rather than by rectangles. Another variation is to bend the needle and keep the parallel lines. Surprisingly, Barbier [**12**] gave an ingenious solution to the original Buffon needle problem by bending the needle into a circle and computing the probability that the circle crosses one of the parallel lines! Gnedenko [**5**] (also see [**9**]) showed that we obtain the same solution if the needle is bent into any convex curve. Buffon's Noodle Problem [**7**] is to find the probability of crossing one of the parallels when tossing a wet noodle of fixed length, but which randomly changes shape on each throw!

H. Solomon [**11**] also discusses higher dimensional analogues of these problems. For example, how can the problem be framed if our needle is to be positioned in euclidean m-space partitioned into parallel hyperplanes or into nonoverlappng m-dimensional rectangles? The problem is then to find the probability that a randomly placed vector with norm n lies entirely in one of the cells.

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Tangents to Conics, Eccentrically

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Geometrical notions are abundant in calculus, where one learns how problems involving them can be addressed via the derivative or integral. Interestingly, in the