The minimal value of $E(y)$ must be either at its boundary points $y = \pm\sqrt{3/2}$ or at $y = 0$, $\pm\sqrt{31/32}$ where $dE/dy = 0$. Comparing $E(y)$ at these five values of y, we find the minimum at $y = \pm\sqrt{31/32}$. Solving for the corresponding z on the parabola and x on the paraboloid gives $P_1(0, \sqrt{31/32}, 1/8)$ and $P_2(0, -\sqrt{31/32}, 1/8)$ as the boundary candidates for minimal $D(y, z)$.

The global minimum for distance-squared is then found by comparing the values of $D(y, z)$ at Q_1, Q_2, P_1 and P_2. We find $D = \sqrt{13}/2$ at Q_1 and Q_2 and $D = \sqrt{63}/8$ at P_1 and P_2. Thus the global minimum is at P_1 and P_2 on the 'hidden' boundary $z = 4 - 4y^2$.

The method of Lagrange multipliers gives the same four candidates and also the point $(0,0,4)$ which we found using one of the values for which $dE/dy = 0$. In some cases the Lagrange method is algebraically easier but in many cases the students (and teachers) miss some of the candidates when solving the system of algebraic (often nonlinear) equations.

References

1. G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, 6th ed., Addison-Wesley, 1984, pp. 866–867.
2. C. S. Ogilvy, Exceptional extremum problems, *American Mathematical Monthly* 67 (1960) 270–275; reprinted in *Selected Papers on Calculus*, MAA, 1969, pp. 262–267.
3. H. A. Thurston, So-called exceptional extremum problems, *American Mathematical Monthly* 68 (1961) 650–652; reprinted in *Selected Papers on Calculus*, MAA, 1969, pp. 268–270.

Another Proof of a Familiar Inequality

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Let $y_1, y_2, \ldots, y_n$ be any permutation of the positive numbers $x_1, x_2, \ldots, x_n$. We use a simple inequality involving the logarithmic function to obtain the following familiar proposition:

$$x_1^{x_1} x_2^{x_2} \cdots x_n^{x_n} \geq y_1^{x_1} y_2^{x_2} \cdots y_n^{x_n} \tag{1}$$

with equality if and only if $x_i = y_i$ $(i = 1, 2, \ldots, n)$.

The standard proof of (1) uses a not particularly simple induction argument. [See D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, NY, 1970, p. 284.]

If $x > 0$, then

$$x - 1 \geq \ln x, \tag{2}$$

with equality if and only if $x = 1$.

(2) follows immediately from the observation that $f(x) = \ln x - x + 1$ has an absolute maximum at $x = 1$, because $f'(x) = 1/x - 1$ vanishes if and only if $x = 1$, and $f''(x) = -1/x^2$ is negative for all x.

Substituting $x = y_i/x_i$ in (2) gives

$$\frac{y_i}{x_i} - 1 \geq \ln \frac{y_i}{x_i} \qquad (i = 1, 2, \ldots, n). \tag{3}$$

Multiplying by x_i and adding, we get

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} \ln\left(\frac{y_i}{x_i}\right)^{x_i}.$$

Since $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$ it follows that

$$0 \geq \ln\left(\frac{y_1}{x_1}\right)^{x_1}\left(\frac{y_2}{x_2}\right)^{x_2} \cdots \left(\frac{y_n}{x_n}\right)^{x_n} \quad \text{or} \quad 1 \geq \left(\frac{y_1}{x_1}\right)^{x_1}\left(\frac{y_2}{x_2}\right)^{x_2} \cdots \left(\frac{y_n}{x_n}\right)^{x_n}. \tag{4}$$

Furthermore, there is equality in (4) if and only if each of the n substituted values for x is 1: that is $x_i/y_i = 1$ $(i = 1, 2, \ldots, n)$.

(3) can also be used to obtain the following related inequality:

If $y_1, y_2, \ldots, y_n$ is any permutation of the positive numbers $x_1, x_2, \ldots, x_n$, then

$$x_1^{1/y_1} x_2^{1/y_2} \cdots x_n^{1/y_n} \geq y_1^{1/y_1} y_2^{1/y_2} \cdots y_n^{1/y_n} \tag{5}$$

with equality if and only if $x_i = y_i$ $(i = 1, 2, \ldots, n)$.

Multiplying (3) by $1/y_i$ gives

$$\frac{1}{x_i} - \frac{1}{y_i} \geq \ln\left(\frac{y_i}{x_i}\right)^{1/y_i}.$$

Hence,

$$\sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} \frac{1}{y_i} \geq \sum_{i=1}^{n} \ln\left(\frac{y_i}{x_i}\right)^{1/y_i}$$

and since

$$\sum_{i=1}^{n} \frac{1}{x_i} = \sum_{i=1}^{n} \frac{1}{y_i},$$

we obtain

$$0 \geq \sum_{i=1}^{n} \ln\left(\frac{y_i}{x_i}\right)^{1/y_i}.$$

This can be written as

$$1 \geq \left(\frac{y_1}{x_1}\right)^{1/y_1}\left(\frac{y_2}{x_2}\right)^{1/y_2} \cdots \left(\frac{y_n}{x_n}\right)^{1/y_n},$$

which gives (5).

Positivity from Evaluation of a Single Point

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The current teaching technique for determining the domain over which a polynomial or a rational expression is positive involves factoring the numerator and the denominator completely, canceling where possible, and evaluating a test point