Thus, $\cosh^{-1}x$ and $\sinh^{-1}x$ are related to the (parametrically determined) hyperbola's sector area A just as $\cos^{-1}x$ and $\sin^{-1}x$ are related to the (parametrically determined) circle's arc length S.

Based on these parametric interpretations, inverse trigonometric functions are called arc functions, and inverse hyperbolic functions are called area functions.

A Self-contained Derivation of the Formula $\frac{d}{dx}(x^r) = rx^{r-1}$ for Rational r

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To establish that $\frac{d}{dx}(x^r) = rx^{r-1}$ for rational r, calculus texts invoke earlier rules of differentiation (quotient rule, chain rule with implicit differentiation, etc.) to first prove this for special cases of r. The purpose of this note is to show that it is possible to establish this result directly, without having to resort to earlier theorems on differentiation. This proof can serve as an instructive exercise for capable students.

Let $r = m/n$, where m and n are positive integers. Then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{m/n} - (x)^{m/n}}{h} = \lim_{h\to 0} \frac{\left[\left\{(x+h)^{1/n}\right\}^m - \left\{x^{1/n}\right\}^m\right]}{\left[\left\{(x+h)^{1/n}\right\}^n - \left\{x^{1/n}\right\}^n\right]}. \qquad (1)$$

By letting $a = (x+h)^{1/n}$ and $b = x^{1/n}$ in the difference formula

$$a^N - b^N = (a-b)(a^{N-1} + a^{N-2}b + \cdots + ab^{N-2} + b^{N-1}),$$

and separately considering $N = m$ and $N = n$, we see that (1) becomes

$$f'(x) = \lim_{h\to 0} \frac{\left\{(x+h)^{1/n} - x^{1/n}\right\} \sum_{i=1}^{m} \left\{(x+h)^{1/n}\right\}^{m-i} \left\{x^{1/n}\right\}^{i-1}}{\left\{(x+h)^{1/n} - x^{1/n}\right\} \sum_{i=1}^{n} \left\{(x+h)^{1/n}\right\}^{n-i} \left\{x^{1/n}\right\}^{i-1}}$$

$$= \frac{\sum_{i=1}^{m} x^{(m-i)/n} \cdot x^{(i-1)/n}}{\sum_{i=1}^{n} x^{(n-i)/n} \cdot x^{(i-1)/n}}.$$

In particular,

$$f'(x) = \frac{\sum_{i=1}^{m} x^{(m-1)/n}}{\sum_{i=1}^{n} x^{(n-1)/n}} = \frac{mx^{(m-1)/n}}{nx^{(n-1)/n}} = (m/n)x^{(m/n)-1}. \qquad (2)$$

Now consider the case of $r = -m/n$, where m and n are positive integers. Here

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{-m/n} - x^{-m/n}}{h} = \lim_{h\to 0} \frac{\{(1/(x+h))^{1/n}\}^m - \{(1/x)^{1/n}\}^m}{h},$$

which can be recast as (3):

$$f'(x) = \lim_{h\to 0} \frac{x^{m/n} - (x+h)^{m/n}}{h(x+h)^{m/n}x^{m/n}} = \lim_{h\to 0} \frac{-1}{(x+h)^{m/n}x^{m/n}} \cdot \lim_{h\to 0} \frac{(x+h)^{m/n} - x^{m/n}}{h}.$$

By definition, the last limiting quotient in (3) is the right-hand expression in (2). Hence,

$$f'(x) = \lim_{h\to 0} \frac{-1}{(x+h)^{m/n} \cdot x^{m/n}} \cdot (m/n)x^{(m/n)-1} = -(m/n)x^{-(m/n)-1}.$$

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Average Values and Linear Functions

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The mean value $M(f;\ a,b) = \frac{1}{b-a}\int_a^b f(t)\,dt$ of an integrable, real-valued function f defined on the interval $[a,b]$ arises frequently in elementary calculus. For example, when f is the continuous instantaneous rate of change (the derivative) of a function g, then (by the fundamental theorem of calculus)

$$M(f;\ a,b) = \frac{1}{b-a}\int_a^b g'(t)\,dt = \frac{g(b)-g(a)}{b-a}$$

is just the average rate of change of g over $[a,b]$. Using the mean value theorem for integrals, one can also show for continuous f that $M(f;\ a,b) = f(c)$ for at least one value of $c \in (a,b)$. As a third illustration, one can express the definite integral of f as the limit of Riemann sums

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(e_i) \cdot \left(\frac{b-a}{n}\right), \qquad e_i = a + i\left(\frac{b-a}{n}\right)$$

and observe that

$$M(f;\ a,b) = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{i=1}^{n} f(e_i);$$

the mean of the function is a limit of "discrete" means.

The purpose of this note is to demonstrate how the mean value of an integrable function can be used to characterize the function's linearity. Quite apart from their intrinsic interest, the arguments below should be helpful to instructors seeking enrichment material that reinforces many of the topics studied in calculus.