- The matrix method produces the solution in a compact form. The solutions produced by computer algebra systems do not offer comparable insight into the structure of the solution without some postprocessing. (Matrix systems of differential and difference equations are discussed in a little book by LaSalle [**4**] that should be read more widely.)
- Our method reinforces the concepts of linearity, matrix multiplication, and the Leibniz rule for differentiating a product, and it provides practice with complex numbers. These are all important topics in science and engineering. The techniques presented here also apply to linear difference equations.
- The idea of differentiating an identity with respect to a parameter, which used to be common in advanced calculus texts, has many applications [**1**]. Thus our method gives students an opportunity to apply important mathematical ideas to solve a class of problems that in the past has often served only to convince students that the introductory differential equations course consists mainly of drill in elementary algebra and calculus.

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The Lighter Side of Differential Equations

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Although differential equations have many serious applications to the modeling of real-world problems, a few lighthearted problems can serve to motivate students and brighten their attitudes toward a computer-oriented course in differential equations. The following two scenarios are uninhibited by reality.

The first problem involves a system of two coupled linear differential equations, which model the ups and downs of a love affair between Romeo and Juliet. In searching for the origins of the basic idea for this problem, we backtracked along an interesting trail and traced the source to Steven Strogatz of MIT. He contributed the problem to a Harvard final examination, although he had originated it during his college days (perhaps when Romeo–Juliet interactions were more compelling). He later wrote a brief article for *Mathematics Magazine* [**7**], and his use of the problem stimulated a column in 1988 by Clarence Peterson in the *Chicago Tribune*, "As Usual, Boy + Girl = Confusion" [**5**]. More recently, Michael Radzicki of Worcester Polytechnic Institute described using a general version of the problem to teach system dynamics skills [**6**]. The problem has surfaced with many variations and is now passing into the folklore. We hope that the following variations are amusing (and original). The lab exercises described were very popular with students at Cornell and Cal Poly and contributed more than any others to the students' understanding of the relationships among the xy, tx, and ty graphs.

Romeo and Juliet (with Apologies to Shakespeare)

Act I. The lovers are haunted by a lack of adjustment in the reciprocity of their feelings toward each other.

> Romeo (cool): "My love for Juliet decreases in proportion to her love for me!"
> Juliet (affectionate): "My love for Romeo grows in proportion to his love for me!"

We will let x represent Romeo's love for Juliet and y represent Juliet's love for Romeo. Time will be measured in days (0–60) and their love will be measured on a scale from -5 to 5, where 0 is indifference:

Hysterical Hatred	Disgust	Indifference	Sweet Attraction	Ecstatic Love
-5	-2.5	0	2.5	5

The equations that model their love affair are

$$\frac{dx}{dt} = -0.2y$$
$$\frac{dy}{dt} = 0.8x$$

where $x =$ R's love for J and $y =$ J's love for R. For initial conditions we will assume that at time $t = 0$, Romeo sees Juliet for the first time and is immediately attracted to her (i.e., $x(0) = 2$). Juliet of course is initially indifferent ($y(0) = 0$); however, the situation is fraught with change! The students will obtain the graphs in Figure 1 [**4**].

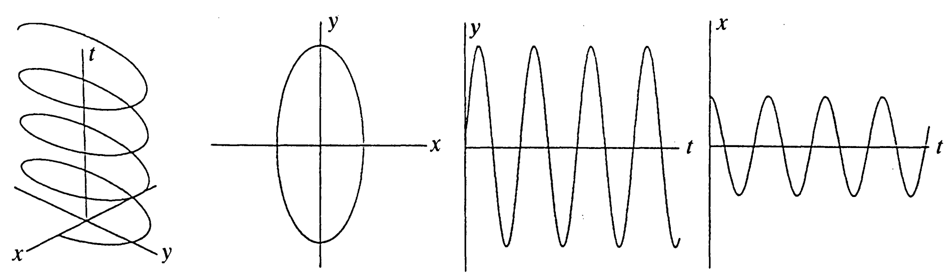

Figure 1
$x' = -0.2y$, $y' = 0.8x$; $-5 \le x \le 5$, $-5 \le y \le 5$, $0 \le t \le 60$.

Interpretation. Juliet experiences far more extreme emotional swings than does Romeo. Beginning to feel dazed from her ride on an emotional rollercoaster, she decides to seek help from a counsellor.

Act II. Deeply affected by her love affair, Juliet is distracted, losing sleep, and forgetting about her homework. The counsellor prescribes a tranquillizer, which tends to decrease Juliet's emotional responses. The tranquillizer's effect is modeled by including another term in the differential equation for Juliet's love for Romeo (i.e., $-0.1y$). The initial conditions and the equation governing Romeo's emotions are assumed to be the same. The new equations are

$$\frac{dx}{dt} = -0.2y \qquad x(0) = 2$$
$$\frac{dy}{dt} = 0.8x - 0.1y \qquad y(0) = 0.$$

The students should obtain the graphs in Figure 2.

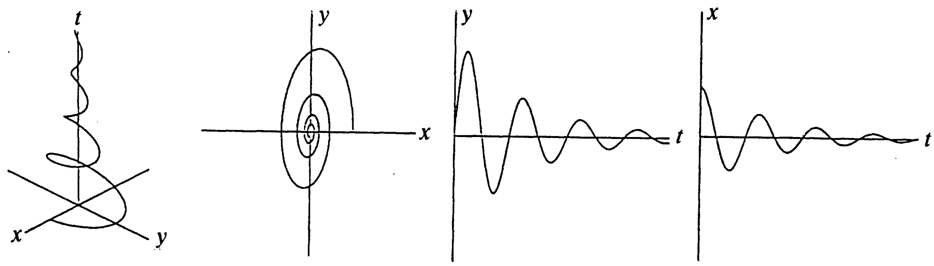

Figure 2
$x' = -0.2y,\ y' = 0.8x - 0.1y;\ -5 \le x \le 5,\ -5 \le y \le 5,\ 0 \le t \le 60.$

Interpretation. The tranquillizer has a damping effect on the entire love affair. Both Romeo and Juliet are appalled by their now insipid relationship. They resolve to seek other advice.

Act III. Juliet stops taking the tranquillizer and together she and Romeo find a new counsellor. They are both sent to therapy to learn to alter their patterns of response to each other. Romeo learns to accept Juliet's love, and now his love begins to decrease only if she becomes overly affectionate ($y > 2$):

$$\frac{dx}{dt} = -0.2(y-2)$$

Juliet learns to control her reactions; her love grows only when Romeo becomes very affectionate ($x > 2$):

$$\frac{dy}{dt} = 0.8(x-2)$$

Assume the same initial conditions. Have they found happiness at last? (See Figure 3.)

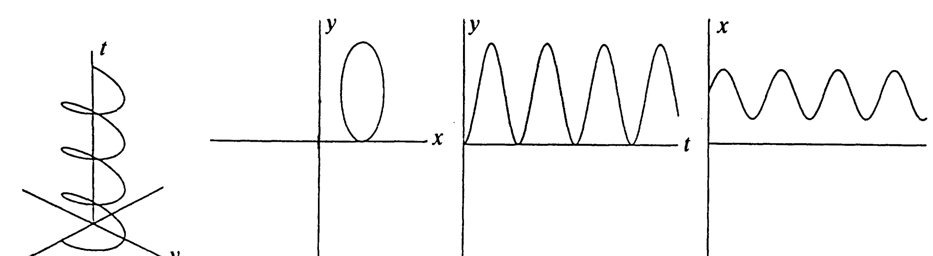

Figure 3
$x' = -0.2(y-2),\ y' = 0.8x(x-2);\ -5 \le x \le 5,\ -5 \le y \le 5,\ 0 \le t \le 60.$

Additional variations in the Romeo–Juliet problem can be found in Strogatz' new book [**8**]. As he mentioned in his article [**7**], there are possibilities for time-dependence (e.g. the onset of spring) or even more intriguing complications: "The term 'many-body problem' takes on a new meaning in this context."

Cannibals

Although of a less literary nature, the second problem has stimulated interest and speculation in computer labs. It involves the competing species model that has been analyzed in several well-known sources [**1**], [**2**]. There is, however, a somewhat new flavor.

First course. Two tribes of cannibals live on an isolated island in the middle of the Indian Ocean. Each tribe is subject to overcrowding but finds the habitat of the other uncomfortable for long stays, so that the overcrowding of one does not affect the overcrowding of the other. Each tribe "dines out" by eating members of the other tribe. In this model, the harvesting of one tribe is viewed as proportional to the number of encounters with members of the other tribe. Also, each tribe cooperates (albeit involuntarily) by increasing the food supply of the other tribe. Let x and y denote the populations (in thousands of individuals) of the first and second tribes, respectively, as functions of time t (in years). The system of equations is given by

Growth	Overcrowding	Cooperation	Harvesting
$\frac{dx}{dt} = 0.4x$	$-0.08x^2$	$+0.02xy$	$-0.06xy$
$\frac{dy}{dt} = 0.2y$	$-0.04y^2$	$+0.03xy$	$-0.05xy$

The graph obtained (Figure 4, page 452) reveals the existence of an equilibrium population, for which $x_e = 3.333$ and $y_e = 3.333$ in thousands [**3**]. This happy balance continues until the arrival of an ecologist from a distant continent.

Second course. The ecologist, concerned by the high overall population of the island as well as the eating habits of the inhabitants, launches a program to educate the natives about the long-term benefits of a low cholesterol diet and the moral imperatives of vegetarianism. He sets up his program with the relatively civilized y tribe, rather than the ferocious x tribe, and convinces them that they should unilaterally refrain from "dining out" in order to set a good example and to start receiving the benefits of lowered cholesterol.

Consequently, in the short run, the growth rates and overcrowding terms show little change, but for the first equation the harvesting term is deleted and in the second equation the cooperation term is deleted. The equations become

$$\frac{dx}{dt} = 0.4x - 0.08x^2 + 0.02xy$$

$$\frac{dy}{dt} = 0.2y - 0.04y^2 - 0.05xy$$

However, the village shaman runs a projection on his new laptop computer and

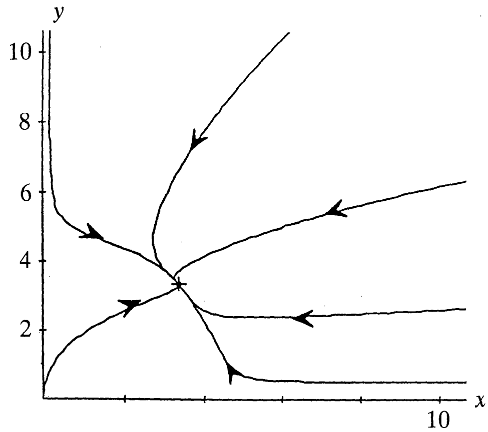

Figure 4
$x' = 0.4x - 0.08x^2 - 0.04xy,$
$y' = 0.2y - 0.04y^2 - 0.02xy.$

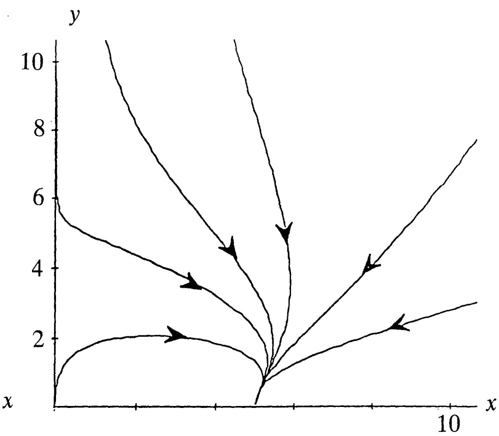

Figure 5
$x' = 0.4x - 0.08x^2 - 0.02xy,$
$y' = 0.2y - 0.04y^2 - 0.05xy.$

obtains the graph in Figure 5. The y tribe after little deliberation decides that low cholesterol in the long run does not compensate for extinction in the short run. They generously invite the x tribe to a feast with the ecologist as guest of honor. (Should he accept?)

Moral. Technology slices both ways!

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Riemann's Postulate

I have tried to avoid long numerical computations, thereby following Riemann's postulate that proofs should be given through ideas and not voluminous computations.

David Hilbert, *Report on Number Theory*, 1897