Now, for differentiable functions f and g, the identity

$$f(x)\,g(x) = \tfrac{1}{2}\left(\left[f(x)+g(x)\right]^2 - f^2(x) - g^2(x)\right)$$

and (*) give

$$\left[f(x)g(x)\right]' = \tfrac{1}{2}\left(2\left[f(x)+g(x)\right]\left[f'(x)+g'(x)\right] - 2f(x)f'(x) - 2g(x)g'(x)\right),$$

which simplifies to

$$\left[f(x)g(x)\right]' = f(x)\,g'(x) + g(x)f'(x).$$

Editor's Note: Once students know that the quotient of differentiable functions is a differentiable function, they may appreciate Marie Agnesi's 1748 proof of the quotient rule: If $h = f/g$, then $hg = f$ and (by the product rule) $hg' + h'g = f$; it remains only to substitute f/g for h and solve for h'.

Angling for Pythagorean Triples

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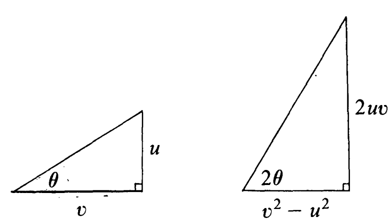
Here is a simple procedure that begins with any proper fraction and produces a Pythagorean triple. To illustrate, begin with a right triangle having an angle θ whose tangent is the given fraction—say $2/3$. Then construct another right triangle using 2θ as one angle. Since $\tan 2\theta = 2\tan\theta/(1-\tan^2\theta) = 12/5$, we may label the legs of the new triangle 12 and 5. The hypotenuse is 13, and $(5, 12, 13)$ is a Pythagorean triple.

We shall see that this procedure always produces Pythagorean triples, and that any Pythagorean triple can be so obtained. Note that the generated triangle depends upon the fraction chosen to express $\tan 2\theta$. If, in the example above, $\tan 2\theta$ had been expressed as $24/10$, the triple $(10, 24, 26)$ would have resulted. Indeed, for any c the triple $(5c, 12c, 13c)$ may be obtained by expressing $\tan 2\theta$ as $5c/12c$. Thus, we are actually producing an acute angle, 2θ, and one representative of a class of similar triangles, rather than a specific Pythagorean triple. For convenience, let us call an acute angle ϕ a *Pythagorean angle* if there is an integer-sided right triangle having ϕ as an angle. (Or, in what amounts to the same thing, ϕ is Pythagorean if $\sin\phi$ and $\cos\phi$ are both rational.) Our procedure may thus be viewed as an algorithm for constructing Pythagorean angles. In particular, we shall prove:

An acute angle 2θ is Pythagorean if and only if $\tan\theta$ *is rational.*

If 2θ is Pythagorean, then $\sin 2\theta$ and $\cos 2\theta$ are rational. Consequently, $\tan\theta = (1-\cos 2\theta)/\sin 2\theta$ is also rational. Conversely, assume that $\tan\theta = u/v$ for positive integers $u < v$. Then $\tan 2\theta = 2uv/(v^2-u^2)$, and the right triangle with legs v^2-u^2 and $2uv$ has hypotenuse u^2+v^2. Thus, 2θ is Pythagorean.



It should be noted that $2uv/(v^2 - u^2)$ will not always be in lowest terms, even when u and v are relatively prime (consider the situation when both are odd). However, one may always express $\tan 2\theta$ in lowest terms as b/a, and the resulting Pythagorean triple (a, b, c) will be *primitive*, that is, a, b and c will have no common factors. Any nonprimitive Pythagorean triple may then be obtained as an integral multiple of a primitive one.

Alert students will quickly observe that two different proper fractions can generate the same triangle. For example, the fractions $1/2$ and $1/3$ both produce a 3-4-5 triangle. This can be explained as follows. Given an integer-sided right triangle, one may choose either acute angle to be 2θ, and thereby determine two different angles θ with distinct rational tangents. Choosing either rational number as the starting point (and, if necessary, reducing the sides of the generated triangle) reproduces the original triangle. It is a nice exercise to find a way to transform the fraction a/b into another fraction that generates the same Pythagorean triple.

Students may find it instructive to relate our approach to producing Pythagorean triples with properties of complex integers $z = a + ib$, where a and b are integral. Clearly, (a, b, c) is a Pythagorean triple if and only if $c = |z|$ is integral. If $w = v + iu$ is a complex integer, then $|w^2| = |w|^2$ is an integer. Therefore, taking $z = w^2$ always produces a Pythagorean triple $(\mathrm{Re}(z), \mathrm{Im}(z), |z|) = (v^2 - u^2, 2uv, v^2 + u^2)$. Thus, one way to produce Pythagorean triples is to square complex integers. Of course, in the complex plane, the squaring operation has the effect of doubling the argument (i.e., the angle that appears in the polar form of a complex number). Accordingly, it is easy to see that our approach is precisely the geometric form of squaring complex integers.

Geometric Parametrization of Pythagorean Triples

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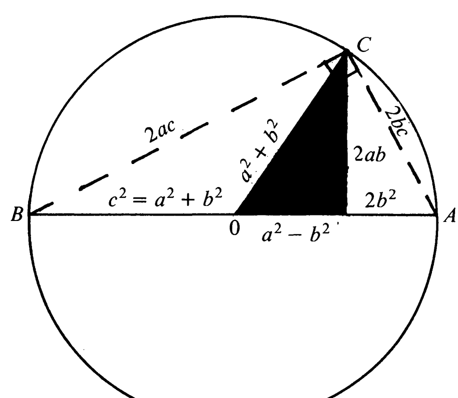
Each Pythagorean triple $a, b, \sqrt{a^2 + b^2}$ generates a new Pythagorean triple $a^2 - b^2$, $2ab, a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + (2ab)^2}$.



Editor's Note: Since $\sphericalangle COA = 2 \sphericalangle B$ and $\sphericalangle COB = 2 \sphericalangle A$, the preceding capsule confirms that the acute angles in the triangle formed by the median and altitude to the hypotenuse of an integer-sided right triangle are Pythagorean.
