An Instant Proof of $e^{\pi} > \pi^{e}$

Norman Schaumberger, Bronx Community College, Bronx, NY

Letting $x = \pi - y$ in

$$e^{x} > \left(1 + \frac{x}{y}\right)^{y}$$

(which holds for x, y both positive) and then taking $y = e$ yields $e^{\pi - e} > (\pi/e)^{e}$, or $e^{\pi} > \pi^{e}$.

Realization of Parity Visits in Walking a Graph

Robert C. Brigham and Ronald D. Dutton, University of Central Florida, Orlando, FL, Phyllis Z. Chinn, Humboldt State University, Arcata, CA, and Frank Harary, University of Michigan, Ann Arbor, MI

In a hamiltonian graph G, there exists a closed walk W which visits each point exactly once. To generalize this phenomenon, we shall show that in any connected graph G there is a walk W_1 which visits each point an odd number of times and another walk W_2 which does this an even number of times. The novelty of our short proof is that it is accomplished by using a standard elementary technique of computer science, the *depth first search*, to obtain a new modest observation in graph theory.

A *walk* in a graph is a sequence of points where consecutive pairs of points are joined by a line in the graph. A *cycle* is a walk with at least three points in which the first and last points are the same and no other point is repeated. A connected graph which contains no cycles is a *tree*. It is well known that every connected graph has a *spanning tree* which is a tree containing all the points of the graph. The tree of Figure 2, for example, is a spanning tree of the graph G in Figure 1. Our graph theoretic terminology follows that of F. Harary's *Graph Theory*, Addison-Wesley, Reading, 1969.

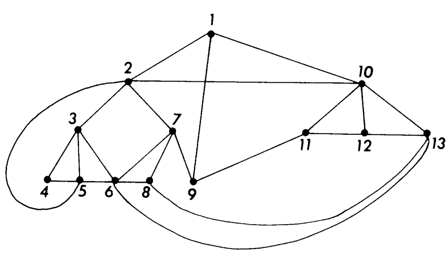

Figure 1. A graph G.

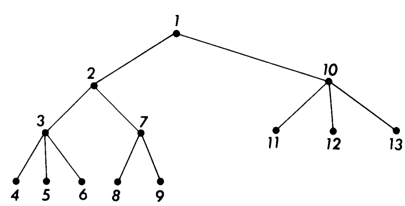

Figure 2. A spanning tree for graph G.

Our main result is illustrated by the following two observations for the tree T of Figure 2:

The walk 1, 2, 3, 4, 3, 5, 3, 6, 3, 2, 3, 2, 7, 8, 7, 9, 7, 2, 1, 2, 1, 10, 11, 10, 12, 10, 13, 10, 1, 10, 1 visits each point an odd number of times.

The walk 2, 3, 4, 3, 4, 3, 5, 3, 5, 3, 6, 3, 6, 3, 2, 3, 2, 7, 8, 7, 8, 7, 9, 7, 9, 7, 2, 7, 2, 1, 10, 11, 10, 11, 10, 12, 10, 12, 10, 13, 10, 13, 10, 1, 10, 1, 2, 1 visits each point an even number of times.

Theorem 1. *In a connected graph G, there is a walk which visits each point an odd (even) number of times.*

To describe an algorithm for generating a walk having the desired property, we introduce the notion of a depth first visitation of a rooted tree. The *sons* of a point v are those points adjacent to v which lie further from the root than v. It is a convention to order the sons of each point from left to right. The *father* of v is the single point adjacent to v which lies closer to the root than v. Thus, in Figure 2, point 3 has sons 4, 5 and 6 and father 2. The root, point 1, has no father. In a depth first visitation, the goal is to move quickly away from the root. The first point visited is the root, followed by the root's leftmost son. In general, the point visited after visiting point v is the leftmost son of v which has not yet been visited. When all sons of v have been visited, the scheme "backs up" to the father of v and continues from there. The visitation halts when it is not possible to back up from the root. The numbering of the points in Figure 2 shows the ordering of the first visit to a point. The walk generated by the scheme is

$$1, 2, 3, 4, 3, 5, 3, 6, 3, 2, 7, 8, 7, 9, 7, 2, 1, 10, 11, 10, 12, 10, 13, 10, 1.$$

Since G is connected, it has a spanning tree; so we need only prove Theorem 1 for a tree T. Let the root of T be r and institute a depth first visitation of the tree, modified as follows. The walk being sought is determined by the order of the points encountered in this modified visitation. The standard depth first search procedure outlined above is followed until the time to back up from a point v. If v has been visited an odd number of times, continue with the standard procedure. Otherwise, back up to v's father, return to v, back up again to v's father, and then proceed once more in the standard manner. Let w be the last visited son of the root r, and consider the potential back up from w to r. Suppose w has been visited an even number of times. If r has been visited an odd number of times, then rwr completes our walk; if r has been visited an even number of times, then rw completes our walk. Suppose w has been visited an odd number of times. Then our walk is already complete if r has been visited an odd number of times, whereas r completes our walk if r has been visited an even number of times. To obtain a walk which visits each point an even number of times, simply interchange the words "even" and "odd" in the above proof. The two walks immediately following the Figures illustrate the modified algorithm.

Suppose one associates an independent parity $f(v) \in \{\text{even, odd}\}$ with each point v of a connected graph G. We further modify our visitation procedure by replacing the word "odd," as applied to a point x, by "$f(x)$" and "even" by the opposite parity to $f(x)$. In this way, we obtain a walk in G which visits each point a number of times which corresponds to its prescribed parity. In fact, we have the more general result:

Corollary. *Associate with each point v_i of a connected graph G two integers: $b_i \geqslant 0$ and $k_i \geqslant 2$. Then there is a walk in G which, with the possible exception of any one point, visits each point v_i a total of b_i(modulo k_i) times.*

For verification of this, modify the algorithm given in the proof of the theorem as follows: when it is time to back up from a point v_i to its father, alternate between v_i and its father until v_i has been visited a total of b_i(modulo k_i) times. In this manner, every point except possibly the root will be visited correctly. Since any point u can be selected as the root of a spanning tree, u can be used as the one point which may not receive the correct number of visits. Observe that if any point is to be visited

0 or 1 (modulo 2) times, then this point may be designated as the root and the algorithm will produce a walk which visits every point the specified number of times.

An *endline* of a tree T is a line incident with an endpoint of T, i.e., a point of degree one. The tree in Figure 2 has eight endpoints.

Theorem 2. *Given any endline of a tree T and $i \in \{0, 1, 2\}$, there exists a walk which originates at one of the points of the endline and visits every point i(modulo 3) times.*

If T has only two points, the result is immediate. Thus, assume T has at least three points. Let uv be an endline, where u is the endpoint. Embed T in the plane with v as root, u as the leftmost son of v and point w as the rightmost son. Now apply the algorithm of our Corollary. If all points, including v, are visited i(modulo 3) times, we are done. Otherwise, we may assume that the resulting walk W ends in w and visits all points i(modulo 3) times except for v. If v is visited $(i-1)$(modulo 3), then the number of times the walk Wv visits v is i(modulo 3) and we are done. Thus, it remains only to consider the case where the number of times W visits v is $(i+1)$(modulo 3). In this case, the walk $uvuWvu$ constructed from W visits all the points a correct number of times.

A slight modification of the preceding proof shows that of any two adjacent points in a tree, at least one can be used as the root of a tree for which the algorithm of our Corollary will produce a walk which visits each point of the tree i(modulo 3) times.

Habitués of video arcades may recognize the applicability of the preceding results to the game of "Q*Bert."

A Note on Integration by Parts

André L. Yandl, Seattle University, Seattle, WA

The point of a textbook exercise such as evaluating $\int x^n e^{ax}\,dx$ is to illustrate repeated integration by parts. Since this technique can be tedious for $n > 1$, students who have learned integration by parts may appreciate the following approach.

To evaluate $\int x^n e^{ax}\,dx$, assume that the answer is of the form $e^{ax}p(x)$, where $p(x)$ is a polynomial of degree n. Then obtain the coefficients of $p(x)$ by setting $D_x\{e^{ax}p(x)\}$ equal to $x^n e^{ax}$. This approach is not only simpler, it introduces students to a technique (the method of undetermined coefficients) they will encounter again in differential equations courses. We illustrate this approach as follows:

Example. To evaluate $\int x^3 e^{2x}\,dx$, we assume that an antiderivative of $x^3 e^{2x}$ is of the form $e^{2x}(Ax^3 + Bx^2 + Cx + D)$. Then

$$D_x\left\{e^{2x}(Ax^3 + Bx^2 + Cx + D)\right\} = x^3 e^{2x}$$

yields

$$e^{2x}\left\{2Ax^3 + (3A + 2B)x^2 + (2B + 2C)x + (C + 2D)\right\} = x^3 e^{2x}.$$

This identity yields: