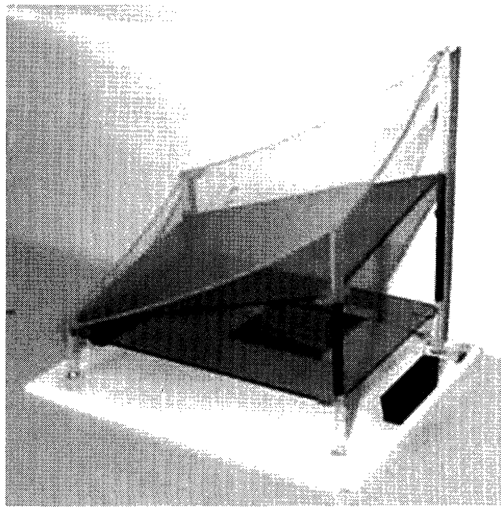

Figure 7. Differentials

to the ground, represents the change of x and the change of y of the function as you move from the front left point of the function. By looking at this model, students can see the physical difference between dz and Δz. They begin to understand that some of the change in z is attributable to the change in x and some of it is attributable to the change in y. Once they can see and grasp these ideas with functions of two variables, they can more easily move to functions of more than two variables.

The model is not difficult to construct from pre-cut plexiglas, available at most home improvement centers. We used sheets that were 8″ by 10″. The biggest challenge in constructing the model was to create a curved surface, but putting a piece of plexiglas in an oven at 500 degrees for 3–4 minutes makes it just soft enough to bend with a little force. After you bend the plexiglas to a shape you are happy with, put it in cold water to cool it and fix the shape.

Because all of the aids mentioned in this article are small, they are most effective in classes of less than fifty students. For large classes, multiple copies can be constructed (these devices are inexpensive and easy to produce). Additional copies would allow teaching assistants to use them in smaller groups to reinforce or clarify difficult concepts. The above are just a few of the ideas we used to help students visualize multivariable concepts. Students repeatedly commented on how useful they were in helping them understand the material rather than just memorizing it. Additionally, instructors enjoyed bringing innovative yet simple "toys" into the classroom.

———o———

Integration from First Principles

Paddy Barry (pdb@ucc.ie) National University of Ireland, Cork, Ireland

The following approach seems to involve an extension of the standard argument for finding from first principles the value of a definite integral. Throughout we suppose that a and b are real numbers satisfying $0 < a < b$ and that

$$\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

is the partition of $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$, with $x_j = a + j\Delta x$ $(0 \le j \le n)$.

Illustrative example. Suppose that f is the function given by $f(x) = \frac{1}{x^2}$ for all $x \neq 0$. We consider $I = \int_a^b \frac{1}{x^2}\,dx$. Then, by definition, I satisfies $s < I < S$, where

$$s = s(\mathcal{P}, f) = \Delta x \sum_{j=0}^{n-1} \frac{1}{x_{j+1}^2} \quad \text{and} \quad S = S(\mathcal{P}, f) = \Delta x \sum_{j=0}^{n-1} \frac{1}{x_j^2}$$

are lower and upper Darboux sums for the Riemann integral I. Clearly $S - s = \Delta x \left(\frac{1}{a^2} - \frac{1}{b^2}\right) \to 0 \quad (n \to \infty)$. Also $0 < I - s < S - s$ so $s \to I \quad (n \to \infty)$, and in turn $S = s + (S - s) \to I \quad (n \to \infty)$. Thus if $(t_n)_{n \in \mathbf{N}}$ is any sequence such that

$$s = s(\mathcal{P}, f) \le t_n \le S(\mathcal{P}, f) = S \quad \text{for all} \quad n \ge 1,$$

by the fly-swatter principle (squeeze principle) we also have that $t_n \to I \quad (n \to \infty)$.

Now here is the key step. Using the observation that

$$\frac{1}{x_{j+1}^2} < \frac{1}{x_j x_{j+1}} < \frac{1}{x_j^2},$$

we make the selection

$$t_n = \Delta x \sum_{j=0}^{n-1} \frac{1}{x_j x_{j+1}} = \sum_{j=0}^{n-1} \left[\frac{1}{x_j} - \frac{1}{x_{j+1}}\right] = \frac{1}{a} - \frac{1}{b},$$

by telescoping. Thus $t_n \to \frac{1}{a} - \frac{1}{b} \quad (n \to \infty)$ and so $I = \frac{1}{a} - \frac{1}{b}$.

Comment. The significant extension in this is that we are not making use of a knowledge in closed form of the values of the upper and lower Darboux sums. Instead, we are choosing an intermediate sum–a telescoping sum–that *can* be expressed in closed form.

Other examples.

1. The method works too for $\int_a^b x^k\,dx$, for positive integers k. Although the extension is not needed here, its use does shorten the calculations. We illustrate for the case $k = 2$. Note that as

   $$v^3 - u^3 = (v - u)(v^2 + vu + u^2),$$

   on taking $u = x_j$ and $v = x_{j+1}$, we have

   $$x_{j+1}^3 - x_j^3 = \Delta x \left[x_{j+1}^2 + x_{j+1}x_j + x_j^2\right]$$

   and so

   $$\Delta x \cdot x_j^2 < \frac{1}{3}\left[x_{j+1}^3 - x_j^3\right] < \Delta x \cdot x_{j+1}^2.$$

   By summing, it follows that $s < \frac{1}{3}(b^3 - a^3) < S$ for every n, so $I = \frac{1}{3}(b^3 - a^3)$.

2. The method also works for $\int_a^b x^{-\frac{k-1}{k}}\,dx$, for integers $k > 1$. We illustrate this in the case of $k = 3$.

We note that as

$$v - u = \frac{v^3 - u^3}{v^2 + vu + u^2},$$

on taking $u = x_j^{1/3}$ and $v = x_{j+1}^{1/3}$ we have

$$x_{j+1}^{1/3} - x_j^{1/3} = \frac{\Delta x}{x_{j+1}^{2/3} + x_{j+1}^{1/3}x_j^{1/3} + x_j^{2/3}}.$$

It follows that, since $0 < a < b$,

$$\Delta x \cdot \frac{1}{x_{j+1}^{2/3}} < 3\left[x_{j+1}^{1/3} - x_j^{1/3}\right] < \Delta x \cdot \frac{1}{x_j^{2/3}}$$

and $s < 3\left(b^{1/3} - a^{1/3}\right) < S$ for every n.

3. To obtain a further extension we consider $I = \int_a^b x^{1/2}\,dx$. For it we start with

$$v^3 - u^3 = (v^2 - u^2)\frac{v^2 + vu + u^2}{v + u},$$

where $u = x_j^{1/2}$, $v = x_{j+1}^{1/2}$ and so have

$$x_{j+1}^{3/2} - x_j^{3/2} = \Delta x \frac{v^2 + vu + u^2}{v + u}.$$

Note that $0 < u < v$, that

$$\frac{v^2 + vu + u^2}{v + u} - \frac{3}{2}u = \frac{(v - u)(2v + u)}{2(v + u)},$$

$$\frac{3}{2}v - \frac{v^2 + vu + u^2}{v + u} = \frac{(v - u)(v + 2u)}{2(v + u)},$$

and that both of these are positive for $0 < u < v$. It follows that

$$\frac{3}{2}u < \frac{v^2 + vu + u^2}{u + v} < \frac{3}{2}v.$$

Hence

$$\Delta x \cdot x_j^{1/2} < \frac{2}{3}\left[x_{j+1}^{3/2} - x_j^{3/2}\right] < \Delta x \cdot x_{j+1}^{1/2}.$$

It follows that $s < \frac{2}{3}\left(b^{2/3} - a^{2/3}\right) < S$ for every n, so $I = \frac{2}{3}\left(b^{2/3} - a^{2/3}\right)$.

———o———