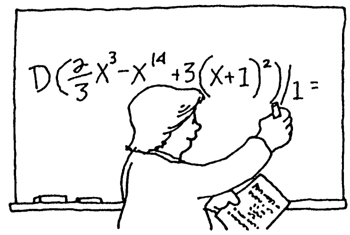
CLASSROOM CAPSULES



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A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

A Note on the Brachistochrone Problem

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As an example of a family of plane curves given by parametric equations, cycloids are described in most calculus textbooks, and it is often mentioned that these curves provide the solution to the brachistochrone (shortest time) problem. Unfortunately, the description of the solution given in many texts is incorrect or misleading.

The brachistochrone problem, originally posed by Johann Bernoulli in 1696, is this: *Given points A and B in a vertical plane, with B below and to the right of A, find a curve joining the points along which a particle might start from rest at A and slide to B in the shortest time, under the assumption that the only force acting on the particle is constant gravity.* (See Figure 1.)

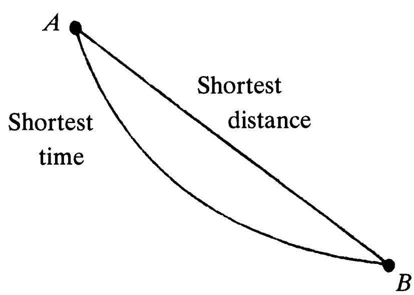

Figure 1

Some conclusions that appear in recent calculus texts are quoted below. Do you notice anything wrong?

- "Among all smooth curves joining [the origin O and the point B at the bottom of the first arch], the cycloid is the curve along which a frictionless bead...will slide from O to B the fastest." [**6**]
- "The correct answer is half of one arch of an inverted cycloid." [**1**]
- "The path that requires the least time coincides with the graph of the inverted cycloid with A at the origin and B the lowest point on the curve." [**5**]

- "It turns out that the way to do this is to construct an inverted cycloid such that B is at the bottom of one of the arches and A is on the arch." [4]

In order to explain the misleading conclusions and to correct them, let's start by choosing coordinates with A at the origin and with the vertical axis pointing *downward*. The incorrect claim that appears most frequently is that the path that requires the least time is half of one arch of a cycloid with its cusp at A and with horizontal tangent at B. That is, B is the bottom point of the arch. But, as we will see, this is correct only when the slope of AB happens to be $2/\pi$. The correct conclusion, due to Johann and Jakob Bernoulli, Newton, and others, is easily described. Based on the *calculus of variations*, the curve to achieve the shortest time is a part of the unique cycloid arch with A at a cusp and with B on the arch. The point B may not be the bottom point of the arch; the location of B on the cycloid arch depends on the slope m of the line AB and there are three distinct cases (Figure 2). To demonstrate that there may be no cycloid with cusp at A and bottom at B, recall that a cycloid $C(k)$ in standard position can be given by parametric equations

$$x = k(\theta - \sin\theta) \qquad \text{and} \qquad y = k(1 - \cos\theta)$$

where k is the radius of the "wheel" that generates the cycloid (Figure 3). One arch is produced by letting θ take on the values from 0 to 2π. The bottom point of this arch occurs when $\theta = \pi$ and has coordinates $(k\pi, 2k)$. Thus, the slope of the line from the cusp at the origin to the bottom point is $2k/k\pi = 2/\pi$.

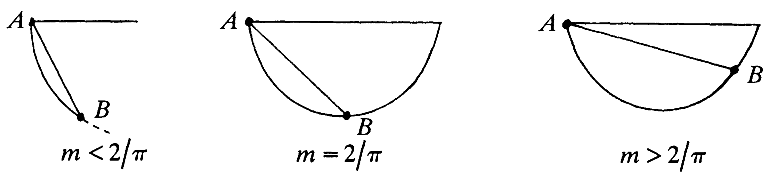

Figure 2

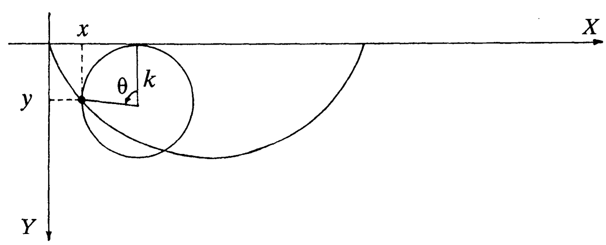

Figure 3

Apparently, any two cycloids are similar; they differ only by a scaling factor. Given that B has positive coordinates (x_1, y_1), it follows that there is exactly one cycloid in standard position with B located on its first arch. This was made clear [7] by Newton with the following geometric argument (Figure 4):

> From the given point A let there be drawn an unlimited straight line $APCZ$ parallel to the horizontal, and on it let there be described an arbitrary cycloid

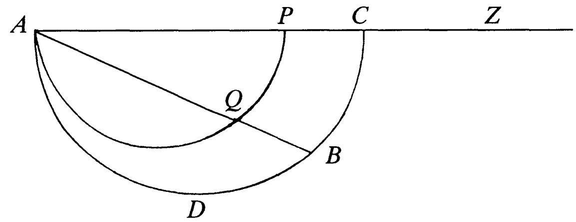

Figure 4

> AQP meeting the straight line AB (assumed drawn and produced if necessary) in the point Q, and further a second cycloid ADC whose base and height are to the base and height of the former as AB to AQ respectively. This last cycloid will pass through the point B, and it will be that curve along which a weight, by the force of its gravity, shall descend most swiftly from the point A to the point B.

The analytic version of this construction is to let $C(k)$ be any cycloid in standard position and let Q be the point where the first arch of $C(k)$ intersects the line AB. The coordinates of Q have the form (rx_1, ry_1) for some positive number r. Now the first arch of $C(k/r)$ meets B because

$$rx_1 = k(\theta - \sin\theta) \qquad \text{implies} \qquad x_1 = (k/r)(\theta - \sin\theta)$$

and similarly for y_1.

For a leisurely exposition of Johann Bernoulli's ingenious solution of the brachistochrone problem, see [3]. A thorough history of the problem may be found in [2].

References

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2. H. Goldstine, *A History of the Calculus of Variations from the 17th through the 19th Century*, Springer-Verlag, New York, 1980.
3. M. Kline, *Calculus: An Intuitive and Physical Approach*, vol. 2, Wiley, New York, 1967, pp. 62–67.
4. A. L. Levine and G. M. Rosenstein, *Discovering Calculus*, McGraw-Hill, New York, 1994, vol. 1, p. 9-50.
5. E. Swokowski, *Calculus with Analytic Geometry*, 3rd ed., PWS-Kent, Boston, 1984, p. 584.
6. G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, 8th ed., Addison-Wesley, Reading, MA, 1992, p. 653.
7. R. Weinstock, Isaac Newton: Credit where credit won't do, *College Mathematics Journal* 25:3 (May 1994) 179–192.

Another Way to Graph a Sequence

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One way to graph a sequence $(a_1, a_2, a_3, \ldots)$ is to plot the points $(1, a_1), (2, a_2), (3, a_3), \ldots$. This common approach emphasizes the fact that the sequence is a function of the counting numbers. Graphically, the beginning of the sequence is emphasized and the tail of the sequence is off the right-hand side of the page (or blackboard). However, the tail is often the part of greatest interest, so an alternative graph that emphasizes the tail is useful.