The Trinomial Triangle

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The familiar Pascal's triangle makes it easy to expand $(a+b)^p$ where p is any positive integer. How about a similar scheme for finding the terms of $(a+b+c)^p$? We will show how to produce a triangular array of coefficients for the terms and then demonstrate why it works. For example we will expand $(a+b+c)^5$. First construct Pascal's triangle from rows 0 to 5 and also list row 5 as a column to the left of the triangle (see Table 1).

Table 1

1	1					
5	1	1				
10	1	2	1			
10	1	3	3	1		
5	1	4	6	4	1	
1	1	5	10	10	5	1

Next, multiply the entries in each row of Pascal's triangle by the number in the far-left column (see Table 2).

Table 2

		n					
		0	1	2	3	4	5
	0	1					
	1	5	5				
m	2	10	20	10			
	3	10	30	30	10		
	4	5	20	30	20	5	
	5	1	5	10	10	5	1

We will call the resulting array of numbers the *trinomial triangle of order* 5. The (m,n) element of this triangle is the coefficient of the term $a^{5-m}b^nc^{m-n}$ in the expansion of $(a+b+c)^5$. For example, the number 30 in row $m=3$ and column $n=2$ corresponds to the term $30a^2b^2c^1$.

The complete expansion, with terms written in positions corresponding to the trinomial triangle is

$$
\begin{array}{llllll}
(a+b+c)^5 = & & & & & \\
a^5 + & & & & & \\
5a^4c + & 5a^4b + & & & & \\
10a^3c^2 + & 20a^3bc + & 10a^3b^2 & & & \\
10a^2c^3 + & 30a^2bc^2 + & 30a^2b^2c + & 10a^2b^3 + & & \\
5ac^4 + & 20abc^3 + & 30ab^2c^2 + & 20ab^3c + & 5ab^4 & \\
c^5 + & 5bc^4 + & 10b^2c^3 + & 10b^3c^2 + & 5b^4c + & b^5
\end{array}
$$

Notice that the powers of a descend with the rows, while the powers of b increase from column to column. The power of c is most easily calculated so that the sum of all three powers is 5. Just as in the binomial theorem, the powers of the variables are easily determined once the pattern is recognized.

To see why the algorithm works, consider the following manipulations. Using the binomial theorem twice we have

$$(a+(b+c))^p = \sum_{m=0}^{p} \binom{p}{m} a^{p-m}(b+c)^m$$
$$= \sum_{m=0}^{p} \binom{p}{m} a^{p-m} \sum_{n=0}^{m} \binom{m}{n} b^n c^{m-n} \quad (1)$$
$$= \sum_{m=0}^{p} \sum_{n=0}^{m} \binom{p}{m}\binom{m}{n} a^{p-m} b^n c^{m-n}. \quad (2)$$

The binomial coefficients $\binom{m}{n}$ multiplying $b^n c^{m-n}$ in (1) are the terms in row m of Pascal's triangle and the factors $\binom{p}{m}$ multiplying a^{p-m} correspond to the column written to the left of Pascal's triangle in Table 1. Thus, the (m, n) elements of the trinomial triangle correspond to the coefficients $\binom{p}{m}\binom{m}{n}$ of the $a^{p-m}b^n c^{m-n}$ term in (2).

The expansion of a trinomial to a power considered in this note is a special case of the so-called multinomial theorem (nicely described in **[1]**):

$$(a_1 + a_2 + \cdots + a_q)^p = \sum_{k_1+k_2+\cdots+k_q=p} \binom{p}{k_1, k_2, \ldots, k_q} a_1^{k_1} a_2^{k_2} \cdots a_q^{k_q},$$

where the multinomial coefficient is calculated as

$$\binom{p}{k_1, k_2, \ldots, k_q} = \frac{p!}{k_1!\, k_2! \cdots k_q!}.$$

It is worth noting that, in the trinomial case $(q = 3)$, the multinomial coefficient can be expressed as

$$\binom{p}{p-m, n, m-n} = \frac{p!}{(p-m)!\, n!(m-n)!} = \frac{p!}{m!(p-m)!} \cdot \frac{m!}{n!(m-n)!}$$
$$= \binom{p}{m}\binom{m}{n},$$

in agreement with the entries in the trinomial triangle.

Reference

1. R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1989, pp. 166–172.