$$\tan\theta \approx \left(\alpha - \frac{5\alpha^3}{192}\right)\left[3\left(1 - \frac{\alpha^2}{16}\right)\right]^{-1}$$

$$\approx \frac{1}{3}\left(\alpha - \frac{5\alpha^3}{192}\right)\left(1 + \frac{\alpha^2}{16}\right)$$

$$\approx \frac{\alpha}{3} + \frac{7\alpha^3}{576} .$$

Finally, by applying Taylor series expansion for arctan we obtain

$$\theta \approx \arctan\left(\frac{\alpha}{3} + \frac{7\alpha^3}{576}\right)$$

$$\approx \frac{\alpha}{3} + \frac{7\alpha^3}{576} - \frac{\alpha^3}{81}$$

$$\approx \frac{\alpha}{3} - \frac{\alpha^3}{5184} .$$

Similar calculations using Peterson's result yields

$$t \approx \frac{\alpha}{3} + \frac{\alpha^3}{648} .$$

Thus, although both θ and t approximate $\alpha/3$, we see that θ gives the better approximation. This could be confirmed by use of a calculator.

———o———

Alternate Approaches to Two Familiar Results

Norman Schaumberger, Bronx Community College, Bronx, NY

It is not particularly easy to prove that $\sqrt[n]{n!}$ becomes infinitely large as n increases or, more exactly,

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e} . \tag{1}$$

The standard proof of (1) involves Stirling's formula (see, for example, A. Taylor, *Advanced Calculus*, Ginn & Co. (1955), p. 684), and so it is rarely included in a first course in calculus.

In this note, we offer a simple proof of (1) that is based on the familiar double inequality

$$\left(1 + \frac{1}{k}\right)^k < e < \left(1 + \frac{1}{k}\right)^{k+1} \tag{2}$$

for all positive integers k. We first give a proof of (2) that rests only on the theorem of the Mean for Derivatives. Thus, (1) and (2) can be presented early in a first course in calculus.

If $f(x)$ is continuous for $a \leqslant x \leqslant b$ and $f'(x)$ exists for $a < x < b$, then there is a number $w \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(w).$$

Let $f(x) = \ln x$, and take $b = k + 1$ and $a = k$. Then the above becomes

$$\ln(k+1) - \ln k = \frac{1}{w} \qquad (k < w < k+1).$$

Hence,

$$\frac{1}{k+1} < \ln(k+1) - \ln k < \frac{1}{k}$$

and it follows that

$$k\left[\ln(k+1) - \ln k\right] < 1 < (k+1)\left[\ln(k+1) - \ln k\right].$$

This can be written as

$$\ln\left(1 + \frac{1}{k}\right)^{k} < 1 < \ln\left(1 + \frac{1}{k}\right)^{k+1}$$

or as (2).

Using (2), we have

$$\prod_{k=1}^{n} \frac{(k+1)^{k+1}}{k^{k+1}} > e^{n} > \prod_{k=1}^{n} \frac{(k+1)^{k}}{k^{k}}.$$

This can be simplified to give

$$\frac{(n+1)^{n+1}}{n!} > e^{n} > \frac{(n+1)^{n}}{n!}.$$

Hence,

$$\frac{(n+1)^{1+1/n}}{e} > (n!)^{1/n} > \frac{n+1}{e}$$

and so

$$\left(\frac{n+1}{n}\right) \cdot \frac{(n+1)^{1/n}}{e} > \frac{(n!)^{1/n}}{n} > \left(\frac{n+1}{n}\right) \cdot \frac{1}{e}.$$

But $(n+1)^{1/n} \to 1$ and $(n+1)/n \to 1$ as $n \to \infty$. Thus, $\lim_{n\to\infty}(n!)^{1/n}/n = 1/e$.