where $D = \text{diag}(1, 2, \ldots, n)$. The answer is

$$f(x) = n!\left(1 + x \sum_{k=1}^{n} \frac{1}{k}\right).$$

The characteristic polynomial. The formula (1) can be directly applied to the characteristic matrix $\lambda I_n - B$ by replacing A by λI_n and B by $-B$. Since $\det(I_n[\alpha|\beta])$ is 0 for $\alpha \neq \beta$ and is 1 for $\alpha = \beta$, we have

$$\det(\lambda I_n - B) = \sum_{r=0}^{n} \sum_{\alpha} \lambda^r \det(-B(\alpha|\alpha))$$
$$= \sum_{r=0}^{n} \lambda^r (-1)^{n-r} b_{n-r} \tag{19}$$

where b_{n-r} is the sum of all $(n-r)$-square principal subdeterminants of B. If $\lambda_1, \ldots, \lambda_n$ are the characteristic roots of B then

$$\det(\lambda I_n - B) = \prod_{i=1}^{n} (\lambda - \lambda_i)$$
$$= \sum_{r=0}^{n} \lambda^r (-1)^{n-r} e_{n-r} \tag{20}$$

where e_k is the kth elementary symmetric polynomial in $\lambda_1, \ldots, \lambda_n$. Matching coefficients in (19) and (20) we have

$$e_k = b_k, \qquad k = 1, \ldots, n.$$

Of course, $k = 1$ and $k = n$ are the familiar

$$\text{tr}(B) = \sum_{i=1}^{n} \lambda_i,$$

and

$$\det(B) = \prod_{i=1}^{n} \lambda_i.$$

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On 'Uniformly Filled' Determinants

Carsten Thomassen and Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA

Let U be a square matrix of order n, and let v be any number. Let $V = (u_{ij} + v)_{i,j=1}^{n}$ be the matrix obtained from U by adding v to each entry of U. In the classroom capsule [**1**] it was observed that

$$\det V = \det U + v \cdot \sum_{i,j=1}^{n} Cof(U)_{ij}, \tag{1}$$

where $Cof(U)$ is the matrix of cofactors of U. The author then applied this result to three examples where U was diagonal.

First, we would like to comment that if U is diagonal the formula can be considerably simplified (thereby responding to the challenge offered by the author at the end of [**1**]). We claim that if $D = \text{diag}(d_1, \ldots, d_n)$, then

$$\det(D + vE) = \left\{1 + v\sum_{j=1}^{n}\frac{1}{d_j}\right\}(d_1 d_2 \cdots d_n), \tag{2}$$

where E is the $n \times n$ matrix of all 1's. (If, say, $d_r = 0$, the right side of (2) reduces to $v(d_1 \cdots d_{r-1}d_{r+1} \cdots d_n)$.)

The three matrices that the author used as examples were all special cases of this formula, at least after multiplying by suitable ± 1's to deal with the fact that those examples used non-principal diagonals.

If we compare (1) and (2), we see that to prove (2) we need only check that the sum of the cofactors of a *diagonal matrix* D is as shown in the second term on the right of (2), which is obvious, since the off-diagonal cofactors all vanish, and the other cofactors are as shown.

Second, we briefly note a generalization of these ideas. Sherman and Morrison [**2**] observed that if A^{-1} is known, and if we now modify A by adding a matrix of rank 1 to it, then there is no need to recompute A^{-1} from the beginning, because the new inverse is

$$(A + \alpha\beta^T)^{-1} = A^{-1} - (A^{-1}\alpha)(\beta^T A^{-1})/(1 + \beta^T A^{-1}\alpha)$$

where α and β are column vectors (this formula is easy to verify simply by multiplying the right side by $A + \alpha\beta^{\mathrm{T}}$ and noting that the identity matrix results).

We remark that a similar formula applies to the determinant of the modified matrix. We claim that the relation

$$\det(U + \alpha\beta^{\mathrm{T}}) = \det U + \beta^T Cof(U)\alpha \tag{3}$$

holds, for every square matrix U and every pair of column vectors α, β.

Indeed, since the determinant is linear, considered as a function of each of its columns, $\det(U + V)$ can be written as a sum of 2^n determinants, each having taken some of its columns from U and the others from V. If V has rank 1, i.e., is of the form $\alpha\beta^{\mathrm{T}}$, then we need consider only those terms in which one or none of the columns are from V, because if two or more columns come from V the determinant will vanish since those two columns will be proportional.

If, say, the jth column is from V (and the others from U) then an expansion by minors down that column gives

$$\sum_i \alpha_i \beta_j Cof(U)_{ij}.$$

If we sum over j, and add in the one term where no columns of V are used, we obtain (3).

There are some interesting special cases.

1°. If U is nonsingular we can write (3) in the form

$$\det(U + \alpha\beta^T) = (1 + \alpha^T U^{-1}\beta)\det U. \tag{4}$$

2°. If in (4) we let $U = I$, we obtain

$$\det\left(I + \alpha\beta^T\right) = 1 + \beta^T\alpha. \tag{5}$$

If we read (5) from right to left, we see an interesting representation of the dot product of two vectors in terms of a determinant.

3°. Finally, in (5) let $\alpha = Ax$, $\beta = x$, where A is a real symmetric matrix and x is a vector. The result is

$$x^TAx = \det\left(I + Axx^T\right) - 1, \tag{6}$$

which shows that every quadratic form in n variables can be represented as a determinant in a simple way.

References

1. Simon M. Goberstein, Evaluating 'uniformly filled' determinants, *The College Mathematics Journal* 19 (1988) 343–345.
2. J. Sherman and W. J. Morrison, Adjustment of an inverse matrix corresponding to changes in the elements of a given column or a given row of the original matrix, *Ann. Math. Stat* 20 (1949) 621.

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At last we see clearly, what mathematicians have claimed for a long time, without being able to present rational grounds, that the differential-*quotient* is the original, the differentials, *dz* and *dy* are derived: .... The thing has taken such a hold of me that it not only goes round my head all day, but last week in a dream I gave a chap my shirtbuttons to differentiate, and he ran off with them.

Friedrich Engels, letter to Karl Marx, August 10, 1881 quoted in *Mathematical Manuscripts of Karl Marx*, 1983, New Park