Binomials to Binomials

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The familiar binomial theorem expands $(a+b)^n$ into a series involving $n+1$ terms.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

The result may then be reduced to the form of a binomial once again, in examples such as

$$\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^9 = 38+17\sqrt{5} \quad \text{and} \quad (2+i)^6 = -117+44i.$$

When the binomial theorem is used to evaluate this last example, first seven terms are calculated, then the four real numbers are added to get -117 and finally the three imaginary numbers are added to get $44i$. In general we write $(a+b)^n = a_n + b_n$, where $a_n = \sum_{k\ even}\binom{n}{k}a^{n-k}b^k$ and $b_n = \sum_{k\ odd}\binom{n}{k}a^{n-k}b^k$. We are looking for a way to recursively generate the terms of these "a and b sequences". Note that $(a+b)^0 = a_0 + b_0 = 1+0$ and $(a+b)^1 = a_1 + b_1 = a+b$ so $\{a_n\} = \{1, a, a_2, a_3, \dots\}$ and $\{b_n\} = \{0, b, b_2, b_3, \dots\}$.

We will describe our method for generating successive terms of the a and b sequences, and then show why it works. Suppose you are given the numbers a and b. Calculate the values $C = 2a$, and $D = b^2 - a^2$. Now successive values of the a and b sequences can be calculated from the recursion relations

$$a_n = Ca_{n-1} + Da_{n-2} \quad \text{and} \quad b_n = Cb_{n-1} + Db_{n-2}. \tag{1}$$

To justify (1) we first observe that

$$a_n = \tfrac{1}{2}\left((a+b)^n + (a-b)^n\right) \quad \text{and} \quad b_n = \tfrac{1}{2}\left((a+b)^n - (a-b)^n\right).$$

This is so because $(-b)^{even} = b^{even}$, while $(-b)^{odd} = -b^{odd}$. Now

$$\begin{aligned}(a+b)^n &= (a+b)^2(a+b)^{n-2} = (a^2+2ab+b^2)(a+b)^{n-2}\\ &= (2a^2+2ab+b^2-a^2)(a+b)^{n-2}\\ &= \left(2a(a+b)+(b^2-a^2)\right)(a+b)^{n-2}\\ &= 2a(a+b)^{n-1} + (b^2-a^2)(a+b)^{n-2}.\end{aligned}$$

In the notation of the a and b sequences, it follows that

$$\begin{aligned}a_n + b_n &= 2a(a_{n-1}+b_{n-1}) + (b^2-a^2)(a_{n-2}+b_{n-2})\\ &= C(a_{n-1}+b_{n-1}) + D(a_{n-2}+b_{n-2})\\ &= (Ca_{n-1}+Da_{n-2}) + (Cb_{n-1}+Db_{n-2}).\end{aligned}$$

In exactly the same way,

$$a_n - b_n = (Ca_{n-1}+Da_{n-2}) - (Cb_{n-1}+Db_{n-2}).$$

Solving for a_n and b_n gives (1).

For example, let us evaluate $(2+i)^6$. Here $a=2$ and $b=i$. So $C=4$ and $D=-5$. Now the recursion relations (1) are

$$-5a_{n-2}+4a_{n-1}=a_n \quad \text{and} \quad -5b_{n-2}+4b_{n-1}=b_n.$$

Since the first two terms of the a sequence are 1 and 2, the next is found to be 3 from the recursion relation. The first two terms of the b sequence are 0 and i, so the next term is $4i$. Continuing in this way we get further terms of the sequences:

$n=0$	1	2	3	4	5	6
$a_n=1$	2	3	2	-7	-38	-117
$b_n=0$	i	$4i$	$11i$	$24i$	$41i$	$44i$

Thus we have all the powers of $2+i$ up to the sixth calculated in succession and

$$(2+i)^6=-117+44i.$$

Notice that the real and imaginary parts are calculated independently. If we only need the real part of $(2+i)^6$ we can ignore the calculation of the b sequence completely.

For another example we show that $\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^9=38+17\sqrt{5}$. Here $a=1/2$ and $b=\sqrt{5}/2$. ($a+b$ is the golden section.) So $2a=C=1$ and $b^2-a^2=D=1$. The recursion relations are particularly nice, being the same as for the Fibonacci sequence:

$$a_{n-2}+a_{n-1}=a_n \quad \text{and} \quad b_{n-2}+b_{n-1}=b_n.$$

The first two terms of the a sequence are 1 and 1/2, so the next is 3/2. The first two terms of the b sequence are 0 and $\sqrt{5}/2$ so the next is $\sqrt{5}/2$. The following terms of the sequences are

$n=0$	1	2	3	4	5	6	7	8	9
$a_n=1$	1/2	3/2	2	7/2	11/2	9	29/2	47/2	38
$b_n=0$	$\frac{\sqrt{5}}{2}$	$\frac{\sqrt{5}}{2}$	$\sqrt{5}$	$\frac{3\sqrt{5}}{2}$	$\frac{5\sqrt{5}}{2}$	$4\sqrt{5}$	$\frac{13\sqrt{5}}{2}$	$\frac{21\sqrt{5}}{2}$	$17\sqrt{5}$

We have easily calculated all the powers of the golden section up to the ninth.

The reader may have observed that both of our examples are of the following type: Let F be a field and let $d\in F$ be chosen so that $\sqrt{d}\notin F$. If α and β are elements of F and n is a positive integer, we want to express $(\alpha+\beta\sqrt{d})^n$ in the form $\alpha_n+\beta_n\sqrt{d}$, where α_n and β_n are in F. Our method applies in all such cases, producing the sequences $a_n=\alpha_n$ and $b_n=\beta_n\sqrt{d}$.

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