## Defining Area in Polar Coordinates

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The standard definition used in teaching students how to find area in polar coordinates is:

*If $r = f(\theta)$ is continuous and nonnegative on $[a, b]$, then the area of the region bounded by $r = f(\theta)$ and $a \le \theta \le b$ is $(\frac{1}{2})\int_a^b r^2\,d\theta$.*

Unfortunately, the definition does not explain how to find the area of a region bounded by $r = f(\theta)$ and $a \le \theta \le b$, where $f(\theta)$ is not defined or is nonpositive as $\theta$ varies from $a$ to $b$. Let's consider an alternate definition:

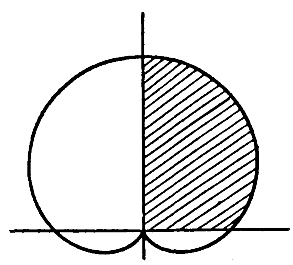
*If $r = f(\theta)$ is continuous on $a \le \theta \le b$, the region bounded by $r = f(\theta)$ as $\theta$ varies from a to b, the radial line containing the point $(f(a), a)$, and the radial line containing the point $(f(b), b)$ is $(\frac{1}{2})\int_a^b r^2\,d\theta$.*

This definition does not require that $r = f(\theta)$ be nonnegative, and it allows that $r = f(\theta)$ may not be defined as $\theta$ varies between the two radial lines which bound the region (it need be defined only where $\theta$ varies in describing the region). It states that the limits of integration are determined by the radial lines between which $\theta$ varies in describing the region, not by the radial lines between which the region lies. If $r = f(\theta)$ is nonnegative on $a \le \theta \le b$, the radial lines between which $\theta$ varies are the radial lines containing the region, and so this alternate definition reduces to the standard one.

The following exercises, which may be difficult using the standard definition, will be easy for those who use the alternate definition.

*Exercise* 1. Find the "first quadrant" area bounded by the graph of $r = \sin\theta - 1$.

| $\theta$ | $\sin\theta - 1$ |
|---|---|
| $0 \to \pi/2$ | $-1 \to 0$ |
| $\pi/2 \to \pi$ | $0 \to -1$ |
| $\pi \to 3\pi/2$ | $-1 \to -2$ |
| $3\pi/2 \to 2\pi$ | $-2 \to -1$ |



$r = \sin\theta - 1$

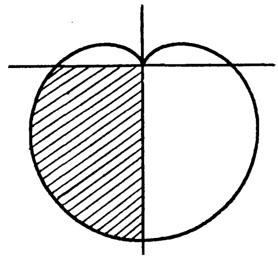
In the region $0 \le \theta \le \pi/2$, we see that $f(\theta)$ is nonpositive. Therefore, students who use the standard definition have to employ additional approaches. For instance:

*Method* 1. Find an equation, having nonnegative values for $r$, which generates the same graph as the given equation. Since $r = \sin\theta + 1$ meets these requirements, the desired area is given by

$$(1/2)\int_0^{\pi/2}(\sin\theta + 1)^2\,d\theta.$$

*Method* 2. Use two facts: (i) The region bounded by $r$ and $a \le \theta \le b$ is congruent to the region bounded by $-r$ and $a + \pi \le \theta \le b + \pi$ (and $a - \pi \le \theta \le b - \pi$) because for any given $\theta$, the points $(f(\theta), \theta)$ and $(-f(\theta), \theta)$ are symmetric with respect to the pole.

(ii) If $r \le 0$ for $a \le \theta \le b$, then $-r \ge 0$ for $a + \pi \le \theta \le b + \pi$ (and $a - \pi \le \theta \le b - \pi$).



$-r$

Based on (i) and (ii), the desired area has value

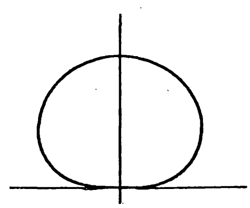
$$(1/2)\int_{\pi}^{3\pi/2}(1 - \sin\theta)^2\, d\theta.$$

Students using the alternate definition need only observe that the region is described as $\theta$ varies from $\pi$ to $3\pi/2$, and is bounded by the radial lines containing $(f(\pi), \pi) = (-1, \pi)$ and $(f(3\pi/2, 3\pi/2) = (-2, 3\pi/2)$. Thus, the area is given by

$$(1/2)\int_{\pi}^{3\pi/2}(\sin\theta - 1)^2\, d\theta.$$

*Exercise* 2. Find the area of the region enclosed by the graph of $r + 2\sqrt{-\sin\theta} = 0$.

| $\theta$ | $-2\sqrt{-\sin\theta}$ |
|---|---|
| $\pi \to 3\pi/2$ | $0 \to -2$ |
| $3\pi/2 \to 2\pi$ | $-2 \to 0$ |



$r + 2\sqrt{-\sin\theta} = 0$

The region lies between $\theta = 0$ and $\theta = \pi$, but $r = f(\theta)$ is not defined on $0 < \theta < \pi$. Therefore, to apply the standard definition, students may seek an equation having nonnegative $f(\theta)$ that generates the same graph as the given equation. Since $r = 2\sqrt{\sin\theta}$ satisfies these requirements, we have

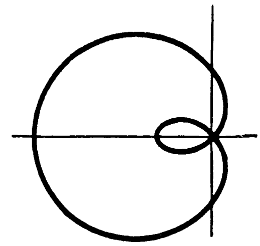
$$(1/2)\int_{0}^{\pi} 4\sin\theta\, d\theta.$$

The alternate definition is more efficient. Since the graph is traced as $\theta$ varies from $\pi$ to $2\pi$, and the points $(0, \pi)$ and $(0, 2\pi)$ lie on the graph, the area is given by

$$(1/2)\int_{\pi}^{2\pi} -4\sin\theta\, d\theta.$$

*Exercise* 3. Find the area of the region inside the small loop of the graph of $r = 1 - 2\cos\theta$.

| $\theta$ | $1 - 2\cos\theta$ |
|---|---|
| $0 \to \pi/3$ | $-1 \to 0$ |
| $\pi/3 \to \pi/2$ | $0 \to 1$ |
| $\pi/2 \to \pi$ | $1 \to 3$ |
| $\pi \to 3\pi/2$ | $3 \to 1$ |
| $3\pi/2 \to 5\pi/3$ | $1 \to 0$ |
| $5\pi/3 \to 2\pi$ | $0 \to -1$ |



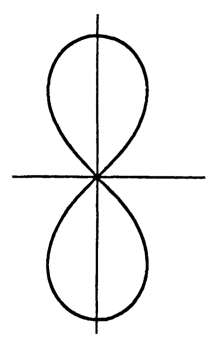
$r = 1 - 2\cos\theta$

Because the standard definition assumes that no radial line crosses the curve more than once between $\theta = a$ and $\theta = b$, it cannot be used for this exercise. With the alternate definition, students easily determine that (because $\theta$ varies from $-\pi/3$ to $\pi/3$ in describing the region, and the equation is satisfied by $(0, -\pi/3)$ and $(0, \pi/3)$) the area is given by

$$(1/2)\int_{-\pi/3}^{\pi/3} (1 - 2\cos\theta)^2\, d\theta.$$

*Exercise* 4. Find the area of the region bounded by one loop of the lemniscate $r^2 + \cos 2\theta = 0$.

| $\theta$ | $\pm\sqrt{-\cos 2\theta}$ |
|---|---|
| $\pi/4 \to \pi/2$ | $0 \to 1$ and $0 \to -1$ |
| $\pi/2 \to 3\pi/4$ | $1 \to 0$ and $-1 \to 0$ |



$r^2 + \cos 2\theta = 0$

Because $f(\theta)$ is neither nonnegative nor nonpositive as $\theta$ varies, the standard definition is not applicable. Now consider the alternate definition. Since both loops are bounded by $r = f(\theta)$ as $\theta$ varies from $\pi/4$ to $3\pi/4$, and one loop contains the points $(0, \pi/4)$ and $(0, 3\pi/4)$, the area bounded by one loop is

$$(1/2)\int_{\pi/4}^{3\pi/4} -\cos 2\theta\, d\theta.$$

Using the alternate definition causes students to realize that the limits of integration are determined by where $\theta$ varies as $r = f(\theta)$ describes the region (not by the location of the region), and that $r = f(\theta)$ may be nonpositive or even undefined as $\theta$ varies between the radial lines that bound the region.