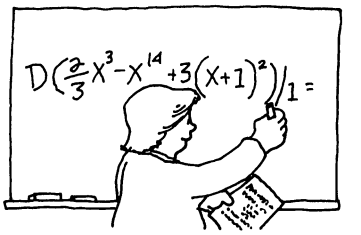
CLASSROOM CAPSULES

EDITORS

Nazanin Azarnia
Department of Mathematics
Santa Fe Community College
Gainesville, FL 32606-6200

Thomas A. Farmer
Department of Mathematics and Statistics
Miami University
Oxford, OH 45056-1641



A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Nazanin Azarnia or Tom Farmer.

The Point-Slope Formula Leads to the Fundamental Theorem of Calculus

Anthony J. Macula, SUNY-Geneseo, Geneseo, NY 14454

The point-slope formula for a line should be familiar to all beginning calculus students. After showing how to compute the average value of a step function, I will show how this simple formula, together with the idea of approximating continuous functions by piecewise linear functions and step functions, leads naturally to the fundamental theorem of calculus. To indicate how easily these ideas can be introduced in the classroom, I present them as a series of "discovery exercises," with commentary. This approach has much in common with an article by Gilbert Strang [Sums and differences vs. integrals and derivatives, *College Mathematics Journal* 21 (1990) 20–27], but where Strang uses velocity and distance, I emphasize slope and area.

Exercise 1. Consider the continuous piecewise linear function F in Figure 1, for which the slopes of the linear segments are shown in Figure 2. If $F(1) = 17.6$, what is $F(4)$?

Solution. Many students would apply the point-slope formula three times to obtain $F(2) = 19.85$, $F(3) = 20.55$, and finally $F(4) = 22.15$. They are impressed when I show them that a simpler way is to compute the average of the local slopes: $\overline{m} = (2.25 + 0.7 + 1.6)/3 = 1.51\overline{6}$, and then use this single average slope in the point-slope formula to find $F(4) = F(1) + \overline{m}(4 - 1) = 22.15$.

Is it true for any continuous piecewise linear function F on $[a, b]$, that

$$F(b) - F(a) = \overline{m}(b - a) \tag{1}$$

where $\overline{m}$ is the average slope of F on $[a, b]$? Exercise 2 is designed to remind students that they know very well how to compute weighted averages of data values, and Exercise 3 demonstrates that (1) holds if the average slope $\overline{m}$ is taken to be a weighted average of the local slopes, i.e., the slopes of the linear segments of F.

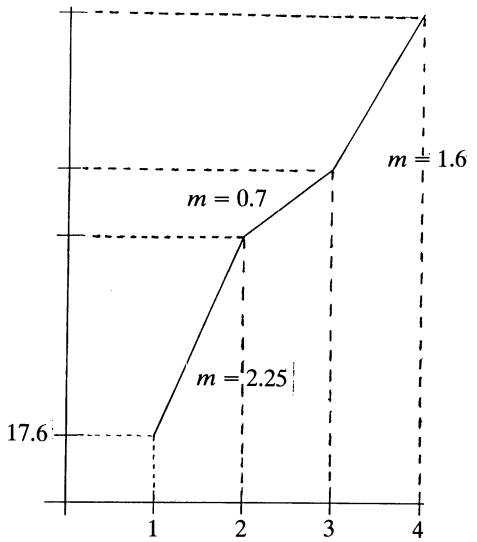


Figure 1
Graph of F.

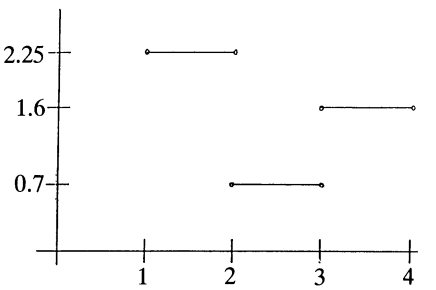


Figure 2
Slope function of F.

Exercise 2. Suppose in Calculus I there were three quizzes plus midterm and final examinations, which carried point totals 30, 40, 80, 120, and 130 respectively, for a total of 400 possible points for the term. The scores on your papers, however, were expressed as percentages: 82%, 72%, 88%, 93%, and 95%. If the professor requires a score of 90% or better for an A, rounding to the nearest percent, would you qualify for an A?

Solution.

$$\text{Term average} = \frac{\text{Points scored}}{\text{Total points}}$$

$$= \frac{(0.82)(30) + (0.72)(40) + (0.88)(80) + (0.93)(120) + (0.95)(130)}{400} = 0.897,$$

which rounds to 90%. Just under the wire!

Exercise 3. Given the continuous piecewise linear function F in Figure 3, with $F(1) = 5.8$ and with local slopes shown in Figure 4, find the average slope $\overline{m}$ and verify that the point-average slope equation (1) gives the correct value for $F(5)$.

Solution. The average slope of F is the weighted average of the local slopes, using as weights the lengths of the subintervals on which F has constant slope:

$$\overline{m} = \frac{(4.0)(0.3) + (1.0)(0.4) + (-1.0)(1.2) + (-2.5)(0.8) + (3.0)(1.3)}{4} = 0.575.$$

The value for $F(5)$ found by using equation (1) can be laboriously verified by using the point-slope formula to successively calculate $F(1.3), F(1.7), \ldots, F(5)$.

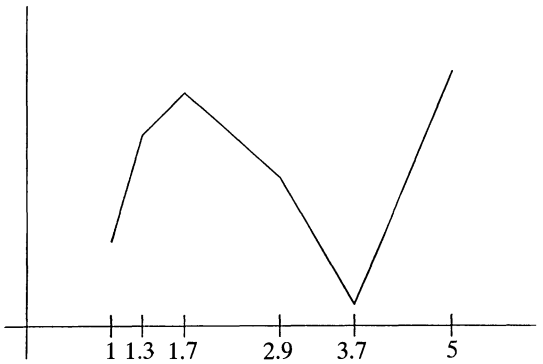


Figure 3
Graph of F.

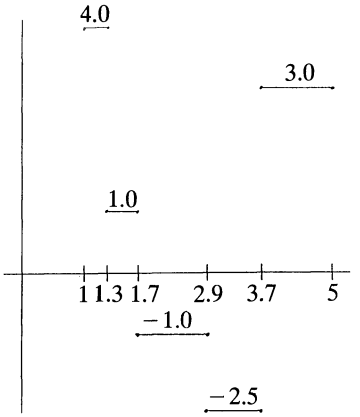


Figure 4
Slope function of F.

Students are pleased with the effort saved by computing a single average slope in exercises like this one. And most have no trouble discovering the geometric interpretation of the average slope, given in the next exercise.

Exercise 4. Connecting the horizontal "steps" in Figures 2 and 4 to the x-axis by vertical lines, we get a collection of rectangular blocks, shown in Figures 5 and 6. Draw the horizontal lines $y = \overline{m}$ on these graphs, using the values of $\overline{m}$ found in Exercises 1 and 3. Show that in each case *the product* $\overline{m}(b-a)$ *gives the area between the graph of the step function and the x-axis, if area below the x-axis is counted as negative.*

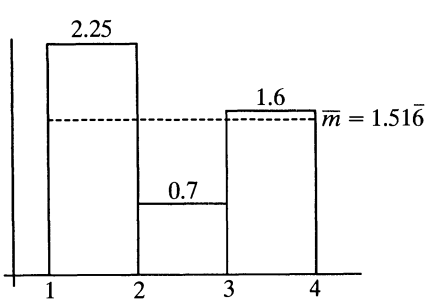


Figure 5

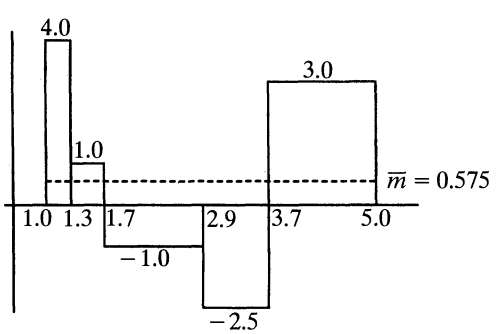


Figure 6

Solution. For the last step, all that is needed is the observation that the numerator in the fraction defining the weighted average $\overline{m}$ is the area A above the x-axis minus the area B below this axis. Since the denominator of this fraction defining $\overline{m}$ is $b-a$, it follows that $\overline{m}(b-a) = A - B$.

When it is pointed out that the step function that gives the local slopes is the *derivative* of the piecewise linear function F, one sees that equation (1) can be

expressed in the language of calculus: *If F is any continuous piecewise linear function on $[a, b]$ and $F' = f$, then*

$$F(b) - F(a) = (\bar{f})(b - a) \tag{2}$$

where $\bar{f}$ is the average value of the step function f on $[a, b]$.

Now it is natural to turn the process around:

Exercise 5. Consider the step function f defined by

$$f(x) = \begin{cases} 4 & \text{if } -1 < x < 0 \\ -2 & \text{if } 0 < x < 0.5 \\ 3 & \text{if } 0.5 < x < 1.3 \\ -1 & \text{if } 1.3 < x < 2.3 \\ 2 & \text{if } 2.3 < x < 4. \end{cases}$$

(i) Calculate $\bar{f}$ and verify that $(\bar{f})(b - a)$ gives the area between the x-axis and the graph of f, counting areas below the axis as negative.

(ii) If F is the continuous piecewise linear antiderivative of $f(x)$ on $[-1, 4]$ with $F(-1) = 3$, use equation (2) to find $F(4)$.

(iii) Find a formula for $F(x)$ on $[-1, 4]$ and verify that the value you found for $F(4)$ is correct.

Solution. As in Exercise 3 we find

$$\bar{f} = \frac{4(1) + (-2)(0.5) + 3(0.8) + (-1)(1) + 2(1.7)}{5} = 1.56.$$

Thus the point-average slope formula (2) gives $F(4) = 3 + (1.56)(4 + 1) = 10.8$. Integrating f "step by step" using the initial value $F(-1) = 3$, or just using the point-slope formula repeatedly, yields

$$F(x) = \begin{cases} 3 + (4)(x + 1) & \text{if } -1 \le x \le 0 \\ 7 + (-2)(x - 0) & \text{if } 0 \le x \le 0.5 \\ 6 + (3)(x - 0.5) & \text{if } 0.5 \le x \le 1.3 \\ 8.4 + (-1)(x - 1.3) & \text{if } 1.3 \le x \le 2.3 \\ 7.4 + (2)(x - 2.3) & \text{if } 2.3 \le x \le 4, \end{cases}$$

which allows us to evaluate $F(4)$ the hard way.

The idea of approximating the graph of a continuous function by a piecewise linear graph is familiar to students who have used computer graphics software such as *Derive*, *Maple*, or *Mathematica*. And students who have used a graphing calculator like the TI Model 81 or 85 should recognize that the graph of any piecewise continuous function can be approximated by a step function, since graphs with a low aspect ratio on a graphing calculator are usually just step functions, as in Figure 7. After a discussion of these two ways to approximate functions, it is natural to ask: Does a piecewise continuous function $f(x)$ on $[a, b]$ have an average value $\bar{f}$ such that the product $(\bar{f})(b - a)$ gives the area between the x-axis and the graph of f, counting areas below the axis as negative, as in

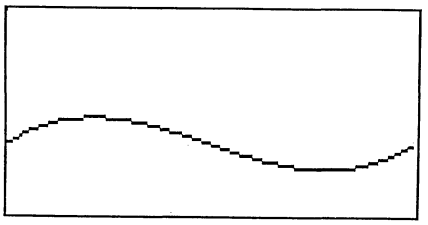


Figure 7

TI-85 graph of $f(x) = (x-2)^3 - x + 3$ on $[1, 3]$.

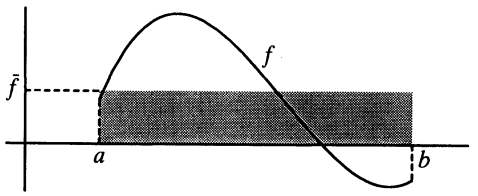


Figure 8

Figure 8? The requirement is that $(\bar{f})(b-a) = \int_a^b f(x)\,dx$, so the answer is YES:

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Now we see that the fundamental theorem of calculus is just a generalization of the point-average slope formula (2), from step functions to arbitrary piecewise continuous functions: *If f is a piecewise continuous function on $[a, b]$ and F is a continuous antiderivative of f on $[a, b]$ then*

$$F(b) - F(a) = (\bar{f})(b-a) = \int_a^b f(x)\,dx. \tag{3}$$

Besides the pedagogical benefit of tying the fundamental theorem of calculus to ideas already familiar to students, this approach has two other merits. First, it introduces the fundamental theorem of calculus for piecewise continuous functions, rather than the more limited case of continuous functions. Exercises with step functions and their continuous piecewise linear antiderivatives, such as Exercise 5, can introduce the idea in a simple setting. Second, it is important for students to learn to picture the derivative as an instantaneous rate of change, a *local average velocity*, not just as the local slope of a graph. Students all know that $\Delta x = v\Delta t$ if the velocity is constant; thus the generalization $\Delta x = \bar{v}\Delta t = \int_{t_0}^{t_1} v(t)\,dt$ for a variable velocity function $v(t)$ is very natural, and this is just our formula (3) in a different setting. This approach to the fundamental theorem of calculus provides an interpretation of integration as *transforming a varying local average $f(x)$ on $[a, b]$ into a global average over this interval*:

$$\bar{f} = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{F(b) - F(a)}{b-a}.$$

This point of view can be helpful in understanding other applications of integrals.

Chebyshev's Theorem: A Geometric Approach

Pat Touhey, College Misericordia, Dallas, PA 18612

Although Chebyshev's theorem is stated in almost all elementary statistics textbooks, few include a proof. The reason is that the usual algebraic proof is not very