row-reduced echelon matrices. Note that

$$7 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2}.$$

In general, the number of distinct type-equivalent $m \times n$ matrices (that is, the number of equivalence classes induced by the "type" relation) is given by

$$N(n,m) = \sum_{k=0}^{\min(m,n)} \binom{n}{k}.$$

To verify this, first observe that the positions of the leading ones in any row-reduced echelon matrix determine the type of *all* the entries in that matrix. (They determine the positions of the forced zeros, and hence those of the undetermined entries.) Thus, all we need show is that the number of ways that the leading ones can be arranged is given by the formula above. Our approach will be to do this for matrices of *rank* k (that is, with k nonzero rows in their reduced form) and then sum the results from rank 0 to rank $\min(m, n)$ (the largest possible).

Suppose A is an $m \times n$ row-reduced echelon matrix of rank k. Then A has exactly k leading ones. These leading ones, located in the first k rows of A, must occur in k distinct columns. Once the columns are specified, the positions of the leading ones are completely determined since they form "stair steps" down to the right. Since there are $\binom{n}{k}$ ways of choosing k objects from a collection of n distinct objects, there are $\binom{n}{k}$ ways of positioning the leading ones. Thus, there are exactly $\binom{n}{k}$ equivalence classes for $m \times n$ row-reduced echelon matrices of rank k. Summing over $k = 0, 1, \ldots, \min(m, n)$ completes the proof.

As an example, observe that the number of distinct type-equivalent 3×3 row-reduced echelon matrices is

$$N(3,3) = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 8.$$

It is no coincidence that the answer turned out to be 2^3. Indeed, the number of distinct type-equivalent square matrices of order n is equal to 2^n. This follows immediately from

$$N(n,n) = \sum_{k=0}^{n} \binom{n}{k},$$

since $(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ yields $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

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Using Riemann Sums in Evaluating a Familiar Limit

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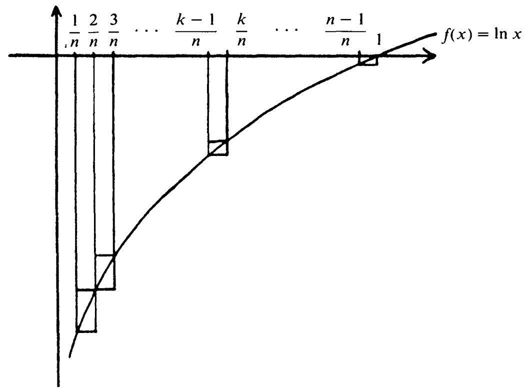
In "Alternate Approaches to Two Familiar Results" [CMJ 15 (November 1984) 422–426], Norman Schaumberger presented an elementary proof that

$$\lim_{n \to \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}. \tag{1}$$

His proof stimulated our thinking and this led to the following geometrically motivated proofs of (1), based on approximating $\int_0^1 \ln x \, dx$ by Riemann sums.



For each $k = 2, 3, \ldots, n$:

$$\frac{1}{n}\ln\left(\frac{k-1}{n}\right) < \int_{(k-1)/n}^{k/n} \ln x\, dx < \frac{1}{n}\ln\left(\frac{k}{n}\right).$$

Hence, on summing over $k = 2, 3, \ldots, n$, we have

$$\ln\left[\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{n-1}{n}\right)\right]^{1/n} < \int_{1/n}^{1} \ln x\, dx < \ln\left[\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\cdots\left(\frac{n}{n}\right)\right]^{1/n}.$$

Integrating and simplifying, we have

$$-1 + \frac{1}{n} < \ln\left[\frac{n!}{n^n}\right]^{1/n} < -1 + \frac{1}{n} + \frac{\ln n}{n}. \tag{2}$$

We can now use L'Hopital's Rule, or the common technique

$$\frac{\ln n}{n} = \left(\frac{\ln\sqrt{n}}{\sqrt{n}}\right)\left(\frac{2}{\sqrt{n}}\right) \leqslant \frac{2}{\sqrt{n}},$$

to show that $(\ln n)/n \to 0$ as $n \to \infty$. Taking the limit in (2) as $n \to \infty$ and using the continuity of exponentiation, we obtain (1).

For a variation on this theme, let $y = (\sqrt[n]{n!}\,)/n$. Then, upon taking natural logs of both sides, we obtain

$$\ln y = (1/n)\left(\sum_{k=1}^{n} \ln k\right) - \ln n = (1/n)\sum_{k=1}^{n} (\ln k - \ln n) = (1/n)\sum_{k=1}^{n} \ln(k/n). \tag{3}$$

The last expression in (3) is a Riemann sum for the area between the x-axis and $y = \ln x$ from $x = 0$ to $x = 1$. Thus,

$$\lim_{n\to\infty} \ln y = \int_0^1 \ln x\, dx = \lim_{a\to 0^+} \int_a^1 \ln x\, dx = \lim_{a\to 0^+} \left[(x\ln x - x)\big|_a^1\right] = -1,$$

and so $\lim_{n\to\infty} y = e^{-1}$ as desired.