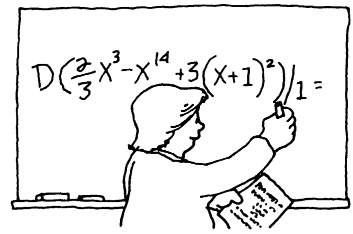
CLASSROOM CAPSULES

EDITOR

Frank Flanigan
Department of Mathematics and Computer Science
San Jose State University
San Jose, CA 95192



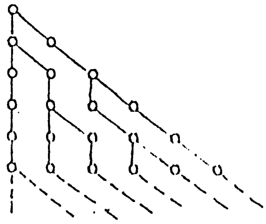
A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Frank Flanigan.

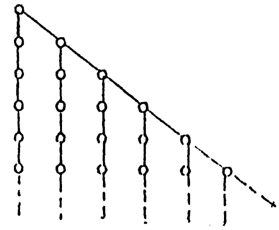
The Number of Paths in a Rooted Binary Tree of Infinite Height

Roger H. Marty, Cleveland State University, Cleveland, OH 44115

Students in discrete mathematics courses are usually skeptical of the fact that there can be an uncountable number of paths (starting at the root) in even the leanest of the rooted binary trees of infinite height. The fact can be illustrated directly using a model of such a tree in terms of intervals on the real line, a setting familiar to students.

Consider a rooted binary tree of countably infinite height such that at every level exactly one of the vertices has two offspring and each remaining vertex has just one offspring. Two such trees are shown below. For every positive integer n, the nth level of such a tree has just $n+1$ vertices. Any such tree is very lean in the sense that (starting at the root) it has only $n+1$ paths of height n for every positive integer n.





To describe a model of a tree that admits uncountably many paths, let $r_1, r_2, \ldots$ denote an enumeration of *all* rational numbers in the open unit interval $I = (0, 1)$. Then for every integer $n \geq 1$, the set $I - \{r_1, r_2, \ldots, r_n\}$ can be thought of as the union of $n+1$ disjoint intervals. (Indeed, if $s_1, s_2, \ldots, s_n$ denotes a reordering of the r's by increasing magnitude, then $I - \{r_1, r_2, \ldots, r_n\} = (0, s_1) \cup (s_1, s_2) \cup \cdots \cup (s_{n-1}, s_n) \cup (s_n, 1)$.)

More formally we define, by mathematical induction, for every integer $n \geq 1$, mutually disjoint open intervals B_i^n $(1 \leq i \leq n+1)$ whose union is $I - \{r_1, r_2, \ldots, r_n\}$ and such that all but one of the intervals B_i^n appear as intervals B_i^{n+1}. The one remaining interval, say B_j^n, contains r_{n+1} and splits into two subintervals determined by r_{n+1}, $B_j^n \cap (0, r_{n+1})$ and $B_j^n \cap (r_{n+1}, 1)$, which become two new intervals in the list B_i^{n+1} $(1 \leq i \leq n+2)$.

In more detail we define $B_1^1 = (0, r_1)$ and $B_2^1 = (r_1, 1)$, which are clearly disjoint open intervals whose union is $I - \{r_1\}$. Assume that intervals B_i^k $(1 \leq i \leq k+1)$ have been defined and satisfy the properties given above. Since one of the B_i^k's, say B_j^k, contains r_{k+1}, we define B_i^{k+1} $(1 \leq i \leq k)$ to coincide in any order with the remaining B_i^k's and define $B_{k+1}^{k+1} = B_j^k \cap (0, r_{k+1})$ and $B_{k+2}^{k+1} = B_j^k \cap (r_{k+1}, 1)$. It follows that the properties given above are satisfied for the B_i^{k+1}'s.

In the model (Figure 1), each vertex in the rooted binary tree is an open interval with rational endpoints and each offspring is a subset (frequently improper) of its parent. Since there are no vertices without offspring in the tree, each path in the (fully developed) tree contains an infinite sequence of nested intervals, $B_{i_1}^1 \supseteq B_{i_2}^2 \supseteq \cdots \supseteq B_{i_n}^n \supseteq \cdots$.

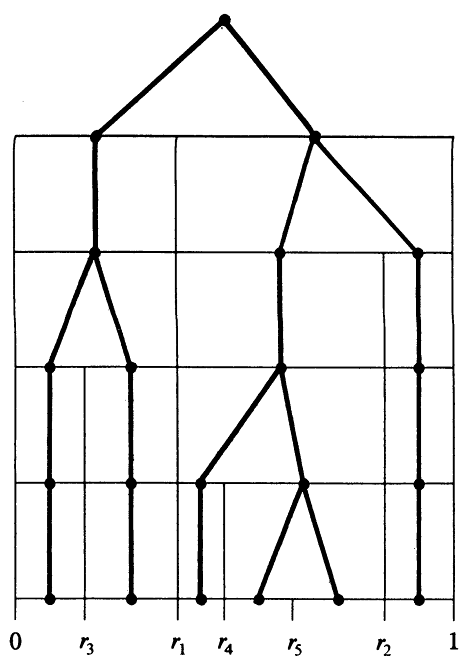


Figure 1

The rooted tree defined by the sequence of rational numbers $r_1, r_2, \ldots$

Define a function f from the Cartesian product of the set of irrational numbers in I and the set of positive integers into the set of positive integers as follows. For every irrational number $p \in I$ and for every $n \geq 1$ there is a unique integer $f(p, n)$ such that $p \in B_{f(p,n)}^n$.

Now define a set-valued function F from the set of irrational numbers in I into the power set of I such that for every irrational number $p \in I$, $F(p) = \bigcap_{n=1}^{\infty} B_{f(p,n)}^n$. The function F is injective for if $F(p) = F(q)$ for irrational numbers $p, q \in I$ with

$p<q$, then for every $n \geq 1$, both $p, q \in B^n_{f(p,n)}$. Consider a rational number r_m such that $p<r_m<q$. Since $p \in B^{m+1}_{f(p,m+1)} \subset (0, r_m)$, it follows that $q \notin B^{m+1}_{f(p,m+1)}$, which is inconsistent with $q \in B^n_{f(p,n)}$ for every $n \geq 1$.

Evidently, $F(p)=\{p\}$ for every irrational number $p \in I$. Since there is an uncountable number of irrational numbers in I and distinct irrational numbers yield distinct paths in the tree, there must be an uncountable number of paths in the tree.

Remarks.

1. If, instead of removing the sequence $\{r_n\}_{n=1}^{\infty}$ of rationals in I, the tree were modeled by removing the sequence of numbers $\{2^{-n}\}_{n=1}^{\infty}$ in I, then there would exist just a countable infinity of paths; namely, for each integer $k \geq 1$ there is a path whose intersection is the interval $(2^{-k}, 2^{-k+1})$, and there is a path whose intersection is the empty set.
2. Similarly, if the terms of the removed sequence—say a_n—are strictly decreasing in magnitude as n increases, then there are just a countable infinity of paths; there are those with intersection (a_k, a_{k-1}) for every integer $k \geq 1$ (with a_0 defined to be 1), and a path whose intersection is the interval $(0, \lim a_k)$, which is possibly degenerate.
3. More generally, suppose the sequence of numbers that are removed from I is eventually decreasing or eventually increasing (i.e., decreasing or increasing on a tail of the sequence or for all terms past some fixed term of the sequence) relative to the usual order on the real line. Then there are only a countable number of paths.

———o———

Tabular Integration by Parts

David Horowitz, Golden West College, Huntington Beach, CA 92647

Only a few contemporary calculus textbooks provide even a cursory presentation of tabular integration by parts [see for example, G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, Addison-Wesley, Reading, MA, 1988]. This is unfortunate because tabular integration by parts is not only a valuable tool for finding integrals but can also be applied to more advanced topics including the derivations of some important theorems in analysis.

The technique of tabular integration allows one to perform successive integrations by parts on integrals of the form

$$\int F(t) G(t)\, dt \tag{1}$$

without becoming bogged down in tedious algebraic details [V. N. Murty, Integration by parts, *Two-Year College Mathematics Journal* 11 (1980) 90–94]. There are several ways to illustrate this method, one of which is diagrammed in Table 1. (We assume throughout that F and G are "smooth" enough to allow repeated differentation and integration, respectively.)