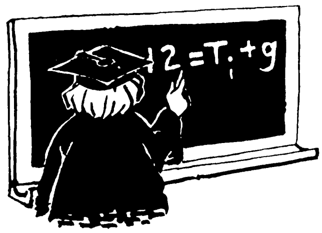

CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

Readers are invited to submit material for consideration to:

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An Algebraic Approach to Partial Fractions

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Separating into partial fractions a rational algebraic function whose denominator has repeated linear or quadratic factors can be quite tedious. In this note, we present an elementary algebraic method for solving such problems. By way of illustration, consider

$$R(x) = \frac{3x+1}{(x-2)(x^2+x+1)^3}.$$

This can be recast as

$$R = R_0 + R_1 + R_2 + R_3,$$

where

$$R_0 = \frac{A}{x-2} \quad \text{and} \quad R_i = \frac{B_i x + C_i}{(x^2+x+1)^i} \qquad (i = 1,2,3).$$

First, we obtain

$$R - R_3 = \frac{-B_3x^2 + (2B_3 - C_3 + 3)x + (2C_3 + 1)}{(x-2)(x^2+x+1)^3}.$$

The numerator of this fraction is divisible by $x^2 + x + 1$. Hence, the remainder $(3B_3 - C_3 + 3)x + (B_3 + 2C_3 + 1)$ is zero. In particular,

$$3B_3 - C_3 = -3 \quad \text{and} \quad B_3 + 2C_3 = -1.$$

Therefore,

$$B_3 = -1 \quad \text{and} \quad C_3 = 0,$$

and

$$R - R_3 = \frac{1}{(x-2)(x^2+x+1)^2}\,.$$

Now, we compute

$$(R - R_3) - R_2 = \frac{-B_2x^2 + (2B_2 - C_2)x + (2C_2+1)}{(x-2)(x^2+x+1)^2}\,.$$

Since $x^2 + x + 1$ divides the numerator of this fraction, the remainder $(3B_2 - C_2)x + (B_2 + 2C_2 + 1)$ is zero. Therefore,

$$3B_2 - C_2 = 0 \quad \text{and} \quad B_2 + 2C_2 = -1,$$

yielding

$$B_2 = -\frac{1}{7} \quad \text{and} \quad C_2 = -\frac{3}{7}\,.$$

Thus,

$$R - R_3 - R_2 = \frac{1}{7(x-2)(x^2+x+1)}\,.$$

The next step is to compute

$$(R - R_3 - R_2) - R_1 = \frac{-7B_1x^2 + (14B_1 - 7C_1)x + (14C_1 + 1)}{7(x-2)(x^2+x+1)}\,.$$

Divide the numerator by $x^2 + x + 1$, and set the remainder $(21B_1 - 7C_1)x + (7B_1 + 14C_1 + 1)$ equal to zero. From

$$3B_1 - C_1 = 0 \quad \text{and} \quad B_1 + 2C_1 = -\frac{1}{7}\,,$$

we find

$$B_1 = -\frac{1}{49} \quad \text{and} \quad C_1 = -\frac{3}{49}\,.$$

The coefficient A may be found independently:

$$A = \lim_{x\to 2}(x-2)R(x) = \frac{1}{49}\,.$$

Taken all together, we obtain

$$\frac{3x+1}{(x-2)(x^2+x+1)^3} = \frac{1}{49(x-2)} - \frac{x+3}{49(x^2+x+1)} - \frac{x+3}{7(x^2+x+1)^2} - \frac{x}{(x^2+x+1)^3}\,.$$