$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}+(-1)^n}=\sum_{n=2}^{\infty}\left\{\frac{(-1)^n}{\sqrt{n}}-\frac{1}{n+(-1)^n\sqrt{n}}\right\}$$

$$=\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}}-\sum_{n=2}^{\infty}\frac{1}{n+(-1)^n\sqrt{n}}, \tag{2}$$

and observing that the first series of (2) converges, whereas the second series diverges because $1/(n+(-1)^n\sqrt{n}) \geqslant 1/(2n)$ for each $n \geqslant 2$.

Hardy uses this example to emphasize the need for a_n to be strictly decreasing to zero as part of the hypothesis for the convergence of an alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$. But this same example can also be used as a counterexample to the extension of the comparison test to alternating series. Indeed, Hardy's divergent series (1) is "dominated" by the convergent alternating series

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}-1},$$

as $1/(\sqrt{n}+(-1)^n) \leqslant 1/(\sqrt{n}-1)$ for all n. Thus, there is no "comparison test" for alternating series.

An example such as this should be presented to all students who study infinite series. Other such illustrations can also be presented—as, for example,

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n(2+(-1)^n)},$$

a divergent alternating series that is dominated by the alternating harmonic series.

Editor's Note: Readers who are interested in this theme may want to refer to R. Lariviere's article "On a Convergence Test for Alternating Series," *Mathematics Magazine* 29 (November-December 1955) 88.

A Note on Differentiation

Russell Euler, Northwest Missouri State University, Maryville, MO

The following technique illustrates an alternate method for deriving the product rule for differentiation.

If $f^2(x)$ is a differentiable function of x, then

$$[f^2(x)]' = \lim_{h\to 0}\frac{f^2(x+h)-f^2(x)}{h}$$

$$= \lim_{h\to 0}\frac{[f(x+h)+f(x)][f(x+h)-f(x)]}{h}$$

$$= 2f(x)f'(x). \tag{*}$$

Now, for differentiable functions f and g, the identity

$$f(x)\,g(x) = \tfrac{1}{2}\left(\left[f(x) + g(x)\right]^2 - f^2(x) - g^2(x)\right)$$

and (∗) give

$$\left[f(x)g(x)\right]' = \tfrac{1}{2}(2\left[f(x) + g(x)\right]\left[f'(x) + g'(x)\right] - 2f(x)f'(x) - 2g(x)g'(x)),$$

which simplifies to

$$\left[f(x)g(x)\right]' = f(x)\,g'(x) + g(x)f'(x).$$

Editor's Note: Once students know that the quotient of differentiable functions is a differentiable function, they may appreciate Marie Agnesi's 1748 proof of the quotient rule: If $h = f/g$, then $hg = f$ and (by the product rule) $hg' + h'g = f$; it remains only to substitute f/g for h and solve for h'.

Angling for Pythagorean Triples

Dan Kalman, Augustana College, Sioux Falls, SD

Here is a simple procedure that begins with any proper fraction and produces a Pythagorean triple. To illustrate, begin with a right triangle having an angle θ whose tangent is the given fraction—say 2/3. Then construct another right triangle using 2θ as one angle. Since $\tan 2\theta = 2\tan\theta/(1 - \tan^2\theta) = 12/5$, we may label the legs of the new triangle 12 and 5. The hypotenuse is 13, and (5, 12, 13) is a Pythagorean triple.

We shall see that this procedure always produces Pythagorean triples, and that any Pythagorean triple can be so obtained. Note that the generated triangle depends upon the fraction chosen to express $\tan 2\theta$. If, in the example above, $\tan 2\theta$ had been expressed as 24/10, the triple (10, 24, 26) would have resulted. Indeed, for any c the triple $(5c, 12c, 13c)$ may be obtained by expressing $\tan 2\theta$ as $5c/12c$. Thus, we are actually producing an acute angle, 2θ, and one representative of a class of similar triangles, rather than a specific Pythagorean triple. For convenience, let us call an acute angle ϕ a *Pythagorean angle* if there is an integer-sided right triangle having ϕ as an angle. (Or, in what amounts to the same thing, ϕ is Pythagorean if $\sin\phi$ and $\cos\phi$ are both rational.) Our procedure may thus be viewed as an algorithm for constructing Pythagorean angles. In particular, we shall prove:

An acute angle 2θ is Pythagorean if and only if $\tan\theta$ is rational.

If 2θ is Pythagorean, then $\sin 2\theta$ and $\cos 2\theta$ are rational. Consequently, $\tan\theta = (1 - \cos 2\theta)/\sin 2\theta$ is also rational. Conversely, assume that $\tan\theta = u/v$ for positive integers $u < v$. Then $\tan 2\theta = 2uv/(v^2 - u^2)$, and the right triangle with legs $v^2 - u^2$ and $2uv$ has hypotenuse $u^2 + v^2$. Thus, 2θ is Pythagorean.