# CLASSROOM CAPSULES

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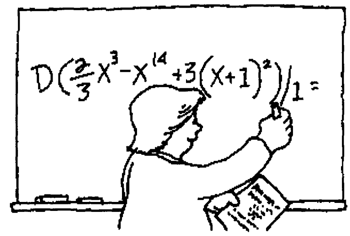


A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

## Bargaining Theory, or Zeno's Used Cars

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Here is a problem that offers an elementary introduction to recursively defined functions. It might be useful in a "mathematics for liberal arts" course, as an easily understood application of basic algebra and the evaluation of a simple limit:

> Billy has a used car for sale and is asking \$2000. Beth offers him \$1500. So Billy splits the difference and asks \$1750. Now Beth splits the difference and offers \$1625. If Billy and Beth continue in ths manner, what common price will they reach? If Beth really wanted to pay \$1600, what should her first offer have been?

Let $p_n$ be the $n$th price offered, so $p_1 = 2000$ and $p_2 = 1500$. Apparently, once the price $p_n$ is on the table, the players arrive at the next price $p_{n+1}$ by averaging $p_n$ and $p_{n-1}$. That is,

$$p_{n+1} = \frac{p_n + p_{n-1}}{2} \tag{1}$$

for $n \geq 2$. This is an example of a recursively defined function (or sequence), and equation (1) can be solved by using the procedure for *second-order, linear, homogeneous recursion relations with constant coefficients* found in some discrete mathematics textbooks (e.g., R. Johnsonbaugh, *Discrete Mathematics,* Macmillan, New York, 1990, pp. 272–279).

However, a more elementary solution exists. Observe that with each bid the difference is cut in half:

$$p_{n+1} - p_n = -\frac{1}{2}(p_n - p_{n-1}).$$

Iterating gives

$$p_{n+1} = p_n + \frac{(-1)^n(p_1 - p_2)}{2^{n-1}}. \tag{2}$$

This is easily proved formally by induction, using equation (1). For if we assume that equation (2) holds for some $n$, then

$$\begin{aligned} p_{n+2} &= \frac{p_{n+1}+p_n}{2} \\ &= \frac{p_{n+1}}{2} + \frac{1}{2}\left(p_{n+1} - \frac{(-1)^n(p_1-p_2)}{2^{n-1}}\right) \\ &= p_{n+1} + \frac{(-1)^{n+1}(p_1-p_2)}{2^n}. \end{aligned}$$

To complete the solution, we iterate equation (2) to get a finite geometric series:

$$\begin{aligned} p_{n+1} &= p_n + \frac{(-1)^n(p_1-p_2)}{2^{n-1}} \\ &= p_{n-1} + \left(\frac{(-1)^{n-1}}{2^{n-1-1}} + \frac{(-1)^n}{2^{n-1}}\right)(p_1-p_2) \\ &\vdots \\ &= p_1 + \left(\frac{(-1)^1}{2^{1-1}} + \cdots + \frac{(-1)^{n-1}}{2^{n-1-1}} + \frac{(-1)^n}{2^{n-1}}\right)(p_1-p_2) \\ &= p_1 - \left[1 + \cdots + (-1/2)^{n-2} + (-1/2)^{n-1}\right](p_1-p_2) \\ &= p_1 - \left(\frac{1-(-1/2)^n}{1-(-1/2)}\right)(p_1-p_2). \end{aligned}$$

Therefore, as $n$ approaches $\infty$, $p_{n+1}$ approaches a limiting value of $p_1-(\frac{2}{3})(p_1-p_2) = \frac{1}{3}p_1 + \frac{2}{3}p_2$, or \$1666.676 in our example.

Finally, we note that if Billy asks initial price $p_1$ and Beth really wants to buy for price $p$, then her first offer $p_2$ should be chosen so that $p = \frac{1}{3}p_1 + \frac{2}{3}p_2$. That is, $p_2 = (3p - p_1)/2$. So since Billy asked \$2000, if Beth wanted to pay \$1600 for the car her first offer should have been \$1400.

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## A Recurrence Relation in the Spinout Puzzle

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Recently I was introduced to a delightful puzzle from William Keister [**3**] called *spinout,* a modern day version of the Chinese Ring Puzzle. While the Chinese Ring Puzzle has been discussed in several places in recent years [**1**], [**2**], spinout is not as well known. It is a nice way to introduce recursion in discrete mathematics or computer science classes.

The spinout puzzle consists of seven gates attached to a moveable bar, which is encased in a frame. The puzzle starts with all of the gates in the vertical position (Figure 1a), and the objective is to turn all the gates one quarter-turn counterclockwise so that the bar with all gates horizontal can be slid to the right out of the frame (Figure 1b). A gate can be turned only when it is above the semicircular opening at the bottom of the frame. The semicircular opening is positioned on the frame so that there is room for one vertical gate to the right of the gate you want to turn. Also, because of their shapes, you cannot turn a gate if the gate to its right is hori-