probability of the response NO, and X_N is the number of NO responses in a random sample of size n, then

$$p_N = P(\text{NO and TAIL}) = \frac{1}{2}(1-p). \tag{3}$$

Solving equation (3) for p, we have $p = 1 - 2p_N$. An estimator of p, $\hat{p}_2$, is given by

$$\hat{p}_2 = 1 - 2\hat{p}_N = 1 - 2\left(\frac{X_N}{n}\right)$$

where X_N has a binomial distribution with mean np_N and variance $np_N(1-p_N)$. As before, the estimator $\hat{p}_2$ is unbiased. Furthermore,

$$\begin{aligned} \text{Var}(\hat{p}_2) &= \frac{4}{n^2}\text{Var}(X_N) = \frac{4}{n^2}np_N(1-p_N) \\ &= \frac{4}{n}\frac{(1-p)}{2}\frac{(1+p)}{2} = \frac{(1-p)+p(1-p)}{n}. \end{aligned} \tag{4}$$

The variance is again larger than would be the case if the question were asked directly.

Comparing $\text{Var}(\hat{p}_1)$ and $\text{Var}(\hat{p}_2)$ given in equations (2) and (4), we see that for $p < 1/4$, $\text{Var}(\hat{p}_1) < \text{Var}(\hat{p}_2)$. In this case, $\hat{p}_1$ is the better estimator of p. For $p > 1/4$, $\hat{p}_2$ is the better estimator. For p = 1/4, the variances of the two estimators are the same.

The normal approximation to the binomial enables us to use the estimators $\hat{p}_1$ and $\hat{p}_2$ to obtain approximate $(1-\alpha)100\%$ confidence intervals for p (see Mendenhall, *Introduction to Probability and Statistics*, 7th ed., Duxbury, Boston, 1987 for a discussion of interval estimates of p). The appropriate formulas for $\hat{p}_1$ and $\hat{p}_2$, respectively, are

$$\hat{p}_1 \pm Z_{\alpha/2}\sqrt{\frac{\frac{3}{4}+\hat{p}_1(1-\hat{p}_1)}{n}}$$

and

$$\hat{p}_2 \pm Z_{\alpha/2}\sqrt{\frac{(1-\hat{p}_2)+\hat{p}_2(1-\hat{p}_2)}{n}}$$

where $Z_{\alpha/2}$ is the value of the standard normal variable with $(\alpha/2)100\%$ of the area under the curve to its right.

Power Series and Exponential Generating Functions

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The following problem originated during a working session with first year undergraduates:

> *Let $P_k(n)$ denote a kth degree polynomial, with real coefficients, in the variable n. Find*
>
> $$\sum_{n=0}^{\infty}\frac{P_k(n)}{n!}x^n. \tag{1}$$

(*Of course it was noticed that* (1) *is everywhere convergent since, by the ratio test, it has an infinite radius of convergence.*)

The solution to this problem yields two complete families of power series (see also Application 1) for which the computation of the sum is possible in quite a nice way. Moreover, this exercise contains some aspects of linear algebra and algorithmic developments, making it an attractive enrichment project for undergraduate students.

As a starting point for summing (1), let $P_k(\mathbb{R})$ denote the $(k+1)$-dimensional real vector space of polynomials of degree less than or equal to k. We take as a basis for $P_k(\mathbb{R})$ the set $\{f_0, f_1, \ldots, f_k\}$, where

$$f_0 = 1, \quad f_1(x) = x, \quad f_2(x) = x(x-1), \ldots, \quad f_k(x) = x(x-1)\cdots(x-k+1).$$

(Students should verify here that this is indeed a basis.) Then $P_k(x)$ can be written as a linear combination of the basis elements

$$P_k(x) = \sum_{i=0}^{k} \lambda_i f_i(x). \tag{2}$$

Let us use this expansion to sum series (1). Specifically,

$$\sum_{n=0}^{\infty} \frac{P_k(n)}{n!} x^n = \sum_{n=0}^{\infty} \frac{\sum_{i=0}^{k} \lambda_i f_i(n)}{n!} x^n = \sum_{i=0}^{k} \lambda_i \left\{ \sum_{n=0}^{\infty} \frac{f_i(n)}{n!} x^n \right\}.$$

(One can ask the students why the Σ's can be interchanged here.)

Now, let us have a look at

$$\sum_{n=0}^{\infty} \frac{f_i(n)}{n!} x^n.$$

For $i = 0$, we have the series representing e^x. For $i \in \{1, 2, \ldots, k\}$, we have $f_i(n) = 0$ for $0 \leq n \leq i-1$. Thus,

$$\sum_{n=0}^{\infty} \frac{f_i(n)}{n!} x^n = \sum_{n=i}^{\infty} \frac{x^n}{(n-i)!} = x^i \sum_{n=i}^{\infty} \frac{x^{n-i}}{(n-i)!} = x^i e^x.$$

So, consideration of $i \in \{0, 1, \ldots, k\}$ yields

$$\sum_{n=0}^{\infty} \frac{P_k(n)}{n!} x^n = \left\{ \sum_{i=0}^{k} \lambda_i x^i \right\} e^x. \tag{3}$$

This means that in order to sum (1), we need only find the coordinates $\{\lambda_0, \lambda_1, \ldots, \lambda_k\}$ of $P_k(n)$ with respect to the basis $\{f_0, \ldots, f_k\}$. This can be done recursively, via (2), by equating coefficients of like powers in the expansions of

$$a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0 = \lambda_0 + \lambda_1 n + \lambda_2 n(n-1) + \cdots + \lambda_k n(n-1)\cdots(n-k+1),$$

or by the algorithm

$$\lambda_0 = P_k(0) \quad \text{and} \quad \lambda_i = \frac{P_k(i) - \sum_{j=0}^{i-1} \lambda_j f_j(i)}{i!} \qquad (i = 1, 2, \ldots, k). \tag{4}$$

Application 1. Series (3) is known in the literature as the exponential generating function for the sequence $\{P_k(n)\colon\ n = 0, 1, 2, \ldots\}$. Therefore, let us consider an application of (3), based on the relation between the ordinary generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ of a sequence $\{a_n\}$ and the exponential generating function

$$E(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

of $\{a_n\}$. The relation

$$A(x) = \int_0^{\infty} e^{-s} E(xs)\, ds \qquad (|x| \le \text{radius of convergence of } A(x)) \tag{5}$$

is readily derived by noting that $n! = \int_0^{\infty} e^{-s} s^n\, ds$, and recasting $A(x)$ as

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n \left(\frac{x^n}{n!} \right) \int_0^{\infty} e^{-s} s^n\, ds = \int_0^{\infty} e^{s} \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} (xs)^n \right) ds.$$

To compute $A(x)$ for the sequence $\{P_k(n)\}$, substitute $E(sx)$ obtained from (3) into (5) (it should be stated that here $A(x)$ converges for $|x| < 1$):

$$\begin{aligned} A(x) &= \int_0^{\infty} \sum_{i=0}^{k} \lambda_i (sx)^i e^{sx} e^{-s}\, ds \qquad (-1 < x < 1) \\ &= \sum_{i=0}^{k} \lambda_i x^i \left(\int_0^{\infty} s^i e^{s(x-1)}\, ds \right). \end{aligned} \tag{6}$$

Now, it is straightforward to verify that

$$\int_0^{\infty} s^i e^{s(x-1)}\, ds = \frac{(-1)^{i+1} i!}{(x-1)^{i+1}},$$

and that (6) therefore can be written as

$$A(x) = \sum_{i=0}^{k} \lambda_i \frac{(-1)^{i+1} i!}{(x-1)^{i+1}} x^i. \tag{7}$$

So again, computation of the λ's is sufficient to sum $A(x) = \sum_{n=0}^{\infty} P_k(n) x^n$.

Example. To sum $\sum_{n=0}^{\infty} (n^2 + 4n) x^n$ (where $|x| < 1$), use (2)

$$n^2 + 4n = 5n + n(n-1) = 0 \cdot f_0(n) + 5 \cdot f_1(n) + f_2(n)$$

to obtain $\lambda_0 = 0$, $\lambda_1 = 5$, $\lambda_2 = 1$. Then (7) yields

$$\sum_{n=0}^{\infty} (n^2 + 4n) x^n = \frac{5x}{(x-1)^2} - \frac{2x^2}{(x-1)^3}.$$

Application 2. As another application of (3), consider the problem of computing higher order derivatives of $(\sum_{i=0}^{k}\lambda_i x^i)e^x$. Using Leibniz's rule, one can make explicit computations in examples; however, finding a general formulation is not easy. Let us use identity (3) as another way of approaching this question. Toward this end, begin with

$$
\begin{aligned}
D^j\left\{\left(\sum_{i=0}^{k}\lambda_i x^i\right)e^x\right\} &= D^j\left\{\sum_{n=0}^{\infty}\frac{P_k(n)}{n!}x^n\right\} \\
&= \sum_{n=j}^{\infty}\frac{P_k(n)}{n!}n(n-1)\cdots(n-j+1)x^{n-j} \\
&= \sum_{n=0}^{\infty}\frac{P_k(n+j)}{(n+j)!}(n+j)\cdots(n+1)x^n \\
&= \sum_{n=0}^{\infty}\frac{P_k(n+j)}{n!}x^n. \qquad (8)
\end{aligned}
$$

Now let's repeat our earlier argument, that began with (2) and ended in (3), for $P_k(n+j)$. In particular, there exist scalars $\{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ such that

$$P_k(n+j) = \sum_{i=0}^{k}\alpha_i f_i(n)$$

and

$$\sum_{n=0}^{\infty}\frac{P_k(n+j)}{n!}x^n = \left(\sum_{i=0}^{k}\alpha_i x^i\right)e^x. \qquad (9)$$

Therefore, (8) and (9) yield

$$D^j\left\{\left(\sum_{i=0}^{k}\lambda_i x^i\right)e^x\right\} = \left(\sum_{i=0}^{k}\alpha_i x^i\right)e^x. \qquad (10)$$

(Note that both polynomial parts have the same degree, since $\lambda_k \neq 0$ implies $\alpha_k \neq 0$.) In other words, to compute the jth derivative of $(\sum_{i=0}^{k}\lambda_i x^i)e^x$ for given $\{\lambda_0, \lambda_1, \ldots, \lambda_k\}$, proceed as follows: *define* $P_k(n) = \sum_{i=0}^{k}\lambda_i f_i(n)$, *express* $P_k(n+j)$ *as* $\sum_{i=0}^{k}\alpha_i f_i(n)$, *and use the computed scalars* $\{\alpha_0, \ldots, \alpha_k\}$ *in* (10).

Example. To compute $D^j(x^k e^x)$, we first observe that $\lambda_k = 1$ and $\lambda_i = 0$ for $i \neq k$. Therefore, $P_k(n) = (1)f_k(n) = n(n-1)\cdots(n-k+1)$, and we seek scalars $\{\alpha_0, \alpha_1, \ldots, \alpha_k\}$ such that $P_k(n+j)$ can be expressed as

$$
\begin{aligned}
&(n+j)(n+j-1)\cdots(n+j-k+1) \\
&\quad = \alpha_0 + \alpha_1 n + \cdots + \alpha_k n(n-1)\cdots(n-k+1). \qquad (11)
\end{aligned}
$$

For small k, this can be solved by equating the coefficients of like powers of n. For large k, we can use a computer to give

$$
\begin{aligned}
\alpha_0 &= P_k(j) \\
\alpha_1 &= P_k(j+1) - P_k(j) \\
&\text{etc.}
\end{aligned}
$$

(In general, $\alpha_0 = D^j\{(\sum_{i=0}^{k}\lambda_i x^i)e^x\}(0) = P_k(j)$ can be seen directly from (9).)

A particular case in point is $k=3$ and $j=5$. Here we easily solve (11),

$$(n+5)(n+4)(n+3)=\alpha_0+\alpha_1 n+\alpha_2 n(n-1)+\alpha_3 n(n-1)(n-2),$$

and obtain

$$D^5(x^3e^x)=(60+60x+15x^2+x^3)e^x.$$

For further details and related applications of (3), see Richard Brualdi's *Introductory Combinatorics*, North Holland, New York, 1983; E. S. Page and L. B. Wilson's *An Introduction to Computational Combinatorics*, Cambridge Computer Science Texts 9, 1979; and John Riordan's *An Introduction to Combinatorial Analysis*, John Wiley & Sons, New York, 1958.

Generalizations of a Complex Number Identity

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A recurring exercise that appears in texts on complex variables is to show that if w and z are complex numbers, then

$$|w|+|z|=\left|(w+z)/2-\sqrt{wz}\right|+\left|(w+z)/2+\sqrt{wz}\right|.$$

In problem 368, this journal, the first author asked for a generalization to three complex numbers. In this note, we give further generalizations to any number of variables and to any dimensional Euclidean space by replacing the complex numbers by vectors.

First, we can simplify the identity by getting rid of the bothersome square roots. Letting $w=z_1^2$ and $z=z_2^2$, we get

$$2\{|z_1|^2+|z_2|^2\}=|z_1-z_2|^2+|z_1+z_2|^2. \tag{1}$$

Geometrically, we now have that the sums of the squares of the edges of a parallelogram equals the sum of the squares of the diagonals. Consequently, by considering a parallelepiped, one generalization is that

$$4\{|z_1|^2+|z_2|^2+|z_3|^2\}=|z_1+z_2+z_3|^2+|z_1+z_2-z_3|^2 + |z_1-z_2+z_3|^2+|-z_1+z_2+z_3|^2. \tag{2}$$

Here, z_1, z_2, z_3 can be complex numbers in the plane or vectors in space. For a proof, assuming the z_i are vectors, just note that

$$|z_1+z_2-z_3|^2=(z_1+z_2-z_3)^2 = z_1^2+z_2^2+z_3^2+2z_1\cdot z_2-2z_1\cdot z_3-2z_2\cdot z_3, \text{ etc.}$$

Geometrically, we have that the sums of the squares of all the edges of a parallelepiped equals the sums of the squares of the four body diagonals. Also to be noted is that (1) is the special case of (2) when $z_3=0$. A generalization to n-dimensional space (for an n-dimensional parallelepiped) is immediate, i.e.,

$$2^n\sum z_i^2=\sum(\pm z_1\pm z_2\pm\cdots\pm z_n)^2, \tag{3}$$