$$\begin{aligned}2A \qquad\qquad\quad &= 1\\ 3A+2B \qquad\quad &= 0\\ 2B+2C \quad &= 0\\ C+2D &= 0.\end{aligned}$$

Since

$$A = 1/2, \qquad B = -3/4, \qquad C = 3/4, \qquad D = -3/8,$$

we have

$$\int x^3e^{2x}\,dx = \left(\tfrac{1}{2}x^3 - \tfrac{3}{4}x^2 + \tfrac{3}{4}x - \tfrac{3}{8}\right)e^{2x} + K.$$

Other integrals, such as $\int e^{ax}\sin bx\,dx$, which require repeated integration by parts can also be evaluated more efficiently using this technique.

Relating Differentiability and Uniform Continuity

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For a continuous function $f\colon R \to R$ define function $F(x) = (f(x) - f(a))/(x - a)$ where a denotes some fixed real number. Clearly F is a continuous function defined on $I - a$, where I is any interval containing a. We wish to prove the following result.

The function $f(x)$ is differentiable at $x = a$ if and only if $F(x)$ is uniformly continuous on some punctured interval $I - a$.

If $f'(a)$ exists, then we may extend $F(x)$ to the continuous function

$$G(x) = \begin{cases} F(x), & x \neq a \\ f'(a), & x = a. \end{cases}$$

Since $G(x)$ is then uniformly continuous on any closed interval I containing a, it follows that $F(x)$ is uniformly continuous on $I - a$.

Suppose, conversely, that $F(x)$ is uniformly continuous on $I - a$, and let $\{x_n\} \in I - a$ be any sequence which converges to a. The sequence $\{x_n\}$ is then a Cauchy sequence and, since $F(x)$ is uniformly continuous, the sequence $\{F(x_n)\}$ is also a Cauchy sequence. By the completeness of the real numbers, there exists a number L such that $F(x_n) \to L$. Furthermore, L does not depend on the choice of $x_n \to a$. Indeed, suppose $\{y_n\} \in I - a$ is another sequence converging to a and let $F(y_n)$ converge to L'. Then the sequence $\{z_n\} \in I - a$ defined by

$$z_n = \begin{cases} x_{(n+1)/2}, & \text{if } n \text{ is odd} \\ y_{n/2}, & \text{if } n \text{ is even} \end{cases}$$

also converges to a, and $L = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} F(z_n) = \lim_{n\to\infty} F(y_n) = L'$. This independence of L on the choice of $x_n \to a$ means that $L = \lim_{x\to a} F(x) = \lim_{x\to a}(f(x) - f(a))/(x - a) = f'(a)$.
