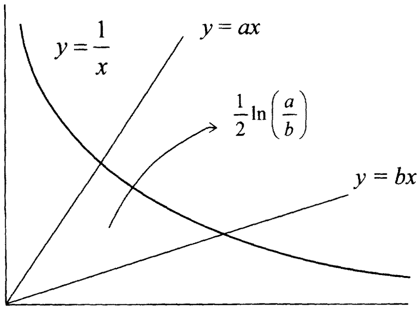

Figure 14.

Acknowledgments. I owe many thanks to Bogdan Mihaila and to Olcay Akman for technical assistance. I thank Prashant Sansgiry for pointing out two related, but different, Proofs Without Words that I include in the references. I don't claim to have thought of the proofs in this article before everybody else, but it certainly felt like that at the time.

References

1. J. Ely, A visual proof that ln(ab) = ln a + ln b, *College Mathematics Journal* **27** (1996) 304.
2. A. H. Stein and D. McGavran, Proof of a common limit, *College Mathematics Journal* **29** (1998) 147.

On A Mean Value Theorem

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Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). Denote by $M = (\frac{a+b}{2}, \frac{f(a)+f(b)}{2})$ the midpoint of the chord from $A = (a, f(a))$ to $B = (b, f(b))$, and let $P = (x, f(x))$ be any point on the graph. See Figure 1.

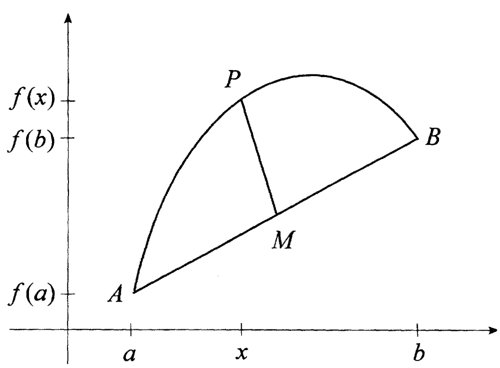

Figure 1.

The distance from M to P is

$$g(x) = \sqrt{\left[x - \frac{a+b}{2}\right]^2 + \left[f(x) - \frac{f(a)+f(b)}{2}\right]^2}, \qquad a \le x \le b,$$

and the values of g at the endpoints are clearly equal (to half the length of the chord AB). We apply Rolle's Theorem to g^2, i.e., to

$$h(x) = \left[x - \frac{a+b}{2}\right]^2 + \left[f(x) - \frac{f(a)+f(b)}{2}\right]^2.$$

Then there is $c \in (a, b)$ for which $h'(c) = 0$. As regards f, this yields the following.

Proposition. *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c)\left[f(c) - \frac{f(a)+f(b)}{2}\right] = -\left[c - \frac{a+b}{2}\right].$$

If $C = (c, f(c))$, and $c \neq (a+b)/2$, this is

(slope of tangent line at C)(slope of MC) $= -1$,

so as with the Mean Value Theorem, this result has an appealing geometric interpretation: Either the line through M and C is perpendicular to the tangent line at C, or M and C coincide.

Notes.

(i) The Cosine Law applied to triangles APM and BPM shows that h is maximized or minimized precisely when the sum of the squares of the chords AP and PB is. This latter quantity is given by the function

$$k(x) = [x-a]^2 + [f(x) - f(a)]^2 + [x-b]^2 + [f(x) - f(b)]^2;$$

Rolle's Theorem applied to it yields another proof of the Proposition.

(ii) By changing slightly the standard auxiliary function used in proving the Mean Value Theorem, J. Tong recently obtained a result similar to the Proposition ([**1, 2**]), the conclusion of which reads (with a different c):

$$f'(c)\left[c - \frac{a+b}{2}\right] = -\left[f(c) - \frac{f(a)+f(b)}{2}\right].$$

Similarly, one may consider $\widehat{k}(x) = [x-a]^2 - [f(x) - f(a)]^2 + [x-b]^2 - [f(x) - f(b)]^2$, for example, to obtain a companion to the Proposition, the conclusion of which reads

$$f'(c)\left[f(c) - \frac{f(a)+f(b)}{2}\right] = \left[c - \frac{a+b}{2}\right].$$

(iii) The standard argument yields an analogue of the Proposition for the Integral Mean Value Theorem: *For continuous f on $[a, b]$, there exists $c \in (a, b)$ such that*

$$f(c)\left[\int_c^b f(t)\,dt - \int_a^c f(t)dt\right] = (c-a) - (b-c).$$

(iv) Other standard arguments now lead to expected analogues of the Proposition and of (iii), for the Cauchy Mean Value Theorems. We leave the interested reader to fill in the details.

References

1. J. Tong, The Mean Value Theorems of Lagrange and Cauchy, *Int. J. Math. Ed. Sci. Tech.* **30** (1999) 456–458.
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Symmetric or Skewed?

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Can the symmetry or skewness of a random variable's distribution be determined solely by inspecting its measures of center? On the other hand, does the direction of skewness indicate the ordering among measures of central tendency? Although there appears to be some confusion in both textbooks and periodicals on these issues, the present note suggests that the answer to both questions—at least with respect to discrete distributions—is "no".

In the past decade, several writers, including Chambers [**1**] and Lee [**2**], have discussed the relationships among measures of central tendency in continuous probability distributions. But as MacGillivray [**3**, p. 366] notes, "the relationship between the mean, medians [sic], and modes for discrete distributions is of course a more difficult problem." Indeed, when discussing discrete distributions, textbooks often make assertions such as, "If the data set is unimodal, but not symmetrical, the mean, mode, and median will be located at different points in the distribution [**6**, p. 47]." This is typically followed by an explicit ordering of the three measures; remarkably, Mogull [**5**] found such presentations in about eighty percent of the textbooks he sampled. Noting that such statements are incorrect, Mogull [**5**, p. 745] argued that "with a positively (negatively) skewed *sample* distribution, both the median and mean lie to the right (left) of the mode but in *unpredictable order*" (emphasis in original). In fact, however, even this weaker claim is invalid. An inequality between the mean and other measures of central tendency may suggest asymmetry in a discrete unimodal distribution, but the reverse is not true. Skewness does not necessarily imply that the mean, median, and mode are unequal. Nor does equality among the measures of central tendency guarantee symmetry in either a discrete probability distribution or a sample distribution.

