Proof by Confrontation. Every college mathematics teacher has seen this at least once. A surly student comes by during office hours and demands an explanation for the grade on his proof. "You took off ten points for that?"

Proof by Frustration. Incomplete results turned in by hard-working students who didn't pick up the "trick" to a proof, like adding and subtracting one to the left side, etc. It is clear that a lot of time was invested in the proof. The student may even write a little note, sharing the agony of his or her struggle with the problem.

Proof by Juxtaposition. Similar to Proof by Bisection, the student, after carefully summarizing the hypotheses, proves an entirely different, perhaps even unrelated, result. While sometimes done accidentally, the usual motivation is for some partial credit on what would otherwise be a total loss.

I would enjoy hearing from readers who have also encountered unusual proofs by —tion.

A Recursively Computed Limit

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Most of today's calculus texts include

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

as either an example or an exercise involving l'Hospital's rule. A few ([**1**], [**2**], [**3**], for instance) also include

$$\lim_{x \to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right)$$

as an exercise—one that leads the students with enough determination to work on it into a morass of algebra and repeated applications of l'Hospital's rule. Even fewer ([**1**], for example) include

$$\lim_{x \to 0^+} \left(\frac{1}{\sin^3 x} - \frac{1}{x^3} \right).$$

The purpose of this note is to present an interesting "recursive" computation of

$$\lim_{x \to 0^+} \left(\frac{1}{\sin^t x} - \frac{1}{x^t} \right),$$

for all real numbers t, using only two straightforward applications of l'Hospital's rule.

To simplify the arguments we will introduce names for certain functions of x and limits as we encounter them. First let

$$F_t(x) = \left(\frac{1}{\sin^t x} - \frac{1}{x^t} \right)$$

and let

$$L_t = \lim_{x \to 0^+} F_t(x).$$

Note that certainly

$$L_t = 0 \qquad (t \le 0). \tag{1}$$

Although (1) is a trivial observation, we will be using that fact for $-1 < t \le 0$ to help compute L_t for $0 < t \le 1$.

Now consider $t > 0$ and observe that

$$F_t(x) = \frac{N_t(x)}{D_t(x)}$$

where

$$N_t(x) = \left(\frac{x}{\sin x}\right)^t - 1 \quad \text{and} \quad D_t(x) = x^t.$$

Since $N_t(x) \to 0$ and $D_t(x) \to 0$ as $x \to 0^+$,

$$L_t = \lim_{x \to 0^+} \frac{N_t(x)}{D_t(x)}$$

is indeterminate. Now

$$\begin{aligned}
\frac{N_t'(x)}{D_t'(x)} &= \frac{t\left(\dfrac{x}{\sin x}\right)^{t-1}\left(\dfrac{\sin x - x\cos x}{\sin^2 x}\right)}{tx^{t-1}} \\
&= \frac{\left\{\left(\dfrac{x}{\sin x}\right)^{t-1} - 1 + 1\right\}\left(\dfrac{\sin x - x\cos x}{\sin^2 x}\right)}{x^{t-1}} \\
&= \frac{\left(\dfrac{x}{\sin x}\right)^{t-1} - 1}{x^{t-1}} \cdot \frac{\sin x - x\cos x}{\sin^2 x} + \frac{\sin x - x\cos x}{x^{t-1}\sin^2 x},
\end{aligned}$$

so that, letting

$$G(x) = \frac{\sin x - x\cos x}{\sin^2 x},$$

we have

$$\frac{N_t'(x)}{D_t'(x)} = F_{t-1}(x) \cdot G(x) + \frac{G(x)}{x^{t-1}}. \tag{2}$$

Let

$$M_t = \lim_{x \to 0^+} \frac{N_t'(x)}{D_t'(x)}.$$

Of course, we want to find M_t which then, by l'Hospital's rule, equals L_t. But first we need to make the following easily verified observations:

$$F_t(x) > 0 \quad \text{for } t > 0 \quad \text{and} \quad 0 < x < \pi, \tag{3}$$

$$G(x) > 0 \quad \text{for } 0 < x < \pi \quad \text{and} \tag{4}$$

$$\lim_{x \to 0^+} \frac{G(x)}{x^{t-1}} = \begin{cases} 0 & \text{if } t < 2 \\ 1/3 & \text{if } t = 2 \\ +\infty & \text{if } t > 2. \end{cases} \tag{5}$$

When verifying (5), first do the case $t = 2$ using l'Hospital's rule and then apply that result to do the other cases. The special case of (5) with $t = 1$ yields

$$\lim_{x \to 0^+} G(x) = 0. \tag{6}$$

Now we are ready to proceed "recursively."
For $0 < t \leq 1$ we have $-1 < t - 1 \leq 0$, so that by (1), (2), (5) and (6)

$$L_t = M_t = 0 \qquad (0 < t \leq 1). \tag{7}$$

For $1 < t < 2$ we have $0 < t - 1 < 1$, so that by (2), (5), (6) and (7)

$$L_t = M_t = 0 \qquad (1 < t < 2).$$

For $t = 2$ we have $t - 1 = 1$, so that by (2), (5), (6) and (7)

$$L_2 = M_2 = 1/3.$$

For $t > 2$ we have $t - 1 > 1$, so that by (3) and (4), for $0 < x < \pi$,

$$F_{t-1}(x) \cdot G(x) + \frac{G(x)}{x^{t-1}} > \frac{G(x)}{x^{t-1}};$$

hence, by (2) and (5),

$$L_t = M_t = +\infty \qquad (t > 2).$$

In summary,

$$\lim_{x \to 0^+} \left(\frac{1}{\sin^t x} - \frac{1}{x^t} \right) = \begin{cases} 0 & \text{if } t < 2 \\ 1/3 & \text{if } t = 2 \\ +\infty & \text{if } t > 2, \end{cases}$$

and we see that the three values L_t attains are represented by the cases $t = 1$, 2 and 3.

References

1. L. Gillman and R. H. McDowell, *Calculus*, 2nd edition, W. W. Norton, New York, 1978.
2. S. L. Salas, E. Hille and J. T. Anderson, *Calculus*, *One and Several Variables*, 5th edition, Wiley, New York, 1986.
3. J. Stewart, *Calculus*, Brooks/Cole, Monterey, CA, 1987.