Doubling: Real, Complex, Quaternion, and Beyond... Well, Maybe

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The division algebra $\mathbb{H}$ of quaternions is commonly understood as 4-tuples of real numbers in the form $a + bi + cj + dk$, with multiplication defined by

$$i^2 = j^2 = k^2 = -1, \qquad ij = k = -ji, \qquad jk = i = -kj, \qquad ki = j = -ik.$$

This definition yields the intimidating expression

$$\begin{aligned}(a_1 + b_1 i + c_1 j + d_1 k)&(a_2 + b_2 i + c_2 j + d_2 k) \\ &= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + a_2b_1 + c_1d_2 - c_2d_1)i \\ &\quad + (a_1c_2 + a_2c_1 + b_2d_1 - b_1d_2)j + (a_1d_2 + a_2d_1 + b_1c_2 - b_2c_1)k.\end{aligned} \tag{1}$$

Perhaps less commonly understood is that $\mathbb{H}$ can be constructed from the field $\mathbb{C}$ of complex numbers in the same way that $\mathbb{C}$ is constructed from the field $\mathbb{R}$ of real numbers, thereby avoiding the unwieldy multiplication (1). In this capsule, we show this construction and how this approach simplifies the proving of quaternion properties.

First, recall how $\mathbb{C}$ is constructed from $\mathbb{R}$. Beginning with a nonreal number i such that $i^2 = -1$, one forms sums $a + bi$, where a and b are real numbers. Addition is defined by

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i \tag{2}$$

and multiplication is defined by

$$(a_1 + b_1 i)(a_2 + b_2 i) = (a_1a_2 - \hat{b}_2 b_1) + (b_2 a_1 + b_1 \hat{a}_2)i, \tag{3}$$

where $\hat{a}$ denotes the (real) conjugate of a, which is simply a itself. Though the use of the conjugate here seems needless, it is very important, as will be seen when we construct the quaternions.

For any complex number $z = a + bi$, the (complex) conjugate of z is defined as

$$\bar{z} = \hat{a} - bi \tag{4}$$

and the norm of z is given by

$$N(z) = z\bar{z} = a\hat{a} + b\hat{b}, \tag{5}$$

or simply $a^2 + b^2$, a sum of two squares.

The following properties of $\mathbb{C}$ are easily established:

(i) multiplication is commutative;
(ii) multiplication is associative;
(iii) $N(z_1 z_2) = N(z_1)N(z_2)$ for $z_1, z_2 \in \mathbb{C}$;
(iv) $z^{-1} = N(z)^{-1}\bar{z}$ if $z \neq 0$.

To construct the quaternions $\mathbb{H}$ from $\mathbb{C}$, we replicate the above procedure, except that the real numbers are replaced with complex numbers. First, define a noncomplex number j such that $j^2 = -1$, and form sums $A + Bj$, where A and B are complex numbers. Define addition by

$$(A_1 + B_1 j) + (A_2 + B_2 j) = (A_1 + A_2) + (B_1 + B_2)j \tag{2'}$$

and multiplication by

$$(A_1 + B_1 j)(A_2 + B_2 j) = (A_1A_2 - \bar{B}_2 B_1) + (B_2 A_1 + B_1 \bar{A}_2)j, \tag{3'}$$

where $\bar{A}$ denotes the (complex) conjugate of A. For $q = A + Bj$, the (quaternion) conjugate of q is

$$q^* = \bar{A} - Bj \tag{4'}$$

and the norm of q is

$$N(q) = qq^* = A\bar{A} + B\bar{B}, \tag{5'}$$

a sum of four squares of real numbers.

Now suppose we let $k = ij$. Then for any $A = a + bi$ and $B = c + di$, we see that $q = a + bi + cj + dk$ is precisely $A + Bj$. In particular,

$$q^* = a - bi - cj - dk \quad \text{and} \quad N(q) = a^2 + b^2 + c^2 + d^2.$$

Taking $A_1 = a_1 + b_1 i$, $B_1 = c_1 + d_1 i$, $A_2 = a_2 + b_2 i$, and $B_2 = c_2 + d_2 i$, and letting $ij = k$, we see that the multiplication (3′) is precisely (1), though in a much simpler form.

Expression (3′) yields the following properties of $\mathbb{H}$ in the same way that (3) yields properties (i)–(iv) for $\mathbb{C}$:

(i′) multiplication is *not* commutative;
(ii′) multiplication is associative;
(iii′) $N(pq) = N(p)N(q)$ for $p, q \in \mathbb{H}$;
(iv′) $q^{-1} = N(q)^{-1}q^*$ if $q \neq 0$.

The reader can quickly convince himself that these properties are more easily obtained from (3′) than from (1)!

Nathan Jacobson (*Basic Algebra I*, W. H. Freeman, San Francisco, 1974, pp. 417–427] shows how to "double" an algebra A by a procedure similar to the one above. Assume that A has unit 1 and that f: $a \to \bar{a}$ is an involution on A such that $a\bar{a} = Q(a)1$, where $Q(a)$ is a non-degenerate quadratic form. We "double" A by forming ordered pairs (a, b) for $a, b \in A$, and by defining addition and multiplication by

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2), \tag{2''}$$

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2 + c\bar{b}_2 b_1, b_2 a_1 + b_1 \bar{a}_2), \tag{3''}$$

where c is any nonzero element of the base field. The resulting algebra B has dimension twice that of A. The involution on B is

$$f_B\colon (a, b) \to \overline{(a, b)} = (\bar{a}, -b) \tag{4''}$$

and the quadratic form is

$$Q_B(a, b) = (a, b)\overline{(a, b)} = (a, b)(\bar{a}, -b) = (a\bar{a} - cb\bar{b})1. \tag{5''}$$

Unfortunately, each doubling results in the loss of some of the structure—as, for example, the loss of commutativity of multiplication when $\mathbb{C}$ is doubled to obtain $\mathbb{H}$. When $\mathbb{H}$ is doubled to obtain the octonions (an 8-dimensional algebra over $\mathbb{R}$), associativity of multiplication is lost. Doubling the octonions is disastrous: properties (i)–(iv) are all lost!

Students often have difficulty appreciating the field properties without seeing some examples where the properties do *not* hold. The examples given here provide some "negative" examples that might be helpful, and the multiplication (3′) is easy to handle.

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