Multiplying by x_i and adding, we get

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} \ln\left(\frac{y_i}{x_i}\right)^{x_i}.$$

Since $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$ it follows that

$$0 \geq \ln\left(\frac{y_1}{x_1}\right)^{x_1}\left(\frac{y_2}{x_2}\right)^{x_2}\cdots\left(\frac{y_n}{x_n}\right)^{x_n} \quad \text{or} \quad 1 \geq \left(\frac{y_1}{x_1}\right)^{x_1}\left(\frac{y_2}{x_2}\right)^{x_2}\cdots\left(\frac{y_n}{x_n}\right)^{x_n}. \quad (4)$$

Furthermore, there is equality in (4) if and only if each of the n substituted values for x is 1: that is $x_i/y_i = 1$ $(i = 1, 2, \ldots, n)$.

(3) can also be used to obtain the following related inequality:

If $y_1, y_2, \ldots, y_n$ is any permutation of the positive numbers $x_1, x_2, \ldots, x_n$, then

$$x_1^{1/y_1} x_2^{1/y_2} \cdots x_n^{1/y_n} \geq y_1^{1/y_1} y_2^{1/y_2} \cdots y_n^{1/y_n} \quad (5)$$

with equality if and only if $x_i = y_i$ $(i = 1, 2, \ldots, n)$.

Multiplying (3) by $1/y_i$ gives

$$\frac{1}{x_i} - \frac{1}{y_i} \geq \ln\left(\frac{y_i}{x_i}\right)^{1/y_i}.$$

Hence,

$$\sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n} \frac{1}{y_i} \geq \sum_{i=1}^{n} \ln\left(\frac{y_i}{x_i}\right)^{1/y_i}$$

and since

$$\sum_{i=1}^{n} \frac{1}{x_i} = \sum_{i=1}^{n} \frac{1}{y_i},$$

we obtain

$$0 \geq \sum_{i=1}^{n} \ln\left(\frac{y_i}{x_i}\right)^{1/y_i}.$$

This can be written as

$$1 \geq \left(\frac{y_1}{x_1}\right)^{1/y_1}\left(\frac{y_2}{x_2}\right)^{1/y_2}\cdots\left(\frac{y_n}{x_n}\right)^{1/y_n},$$

which gives (5).

Positivity from Evaluation of a Single Point

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The current teaching technique for determining the domain over which a polynomial or a rational expression is positive involves factoring the numerator and the denominator completely, canceling where possible, and evaluating a test point between each pair of zeros, from both the numerator and the denominator. This can be laborious, especially if the test point cannot be an integer.

Suppose, for example, a student is asked to determine the domain for which $(6x^2+7x+2)(4x^2-1)/(144x^2-49)$ is positive. The factoring process yields $(3x+2)(2x+1)^2(2x-1)/(12x+7)(12x-7)$, producing $-2/3$, $-1/2$, $1/2$, $-7/12$, and $7/12$ as candidates for sign change. The next step is to order these candidates on a number line and pick a test point between each consecutive pair.

I		II		III		IV		V		VI
	$-2/3$		$-7/12$		$-1/2$		$1/2$		$7/12$	

Fractions cannot be avoided when selecting test points for regions II, III, and V.

This can all be avoided with one simple observation. The rational expression above must change sign where the factor producing the zero has an odd order, while no change in sign can occur if the factor has an even order. Therefore, the expression above must change sign at $-2/3$, $-7/12$, $1/2$, and $7/12$, and at no other value. This reduces the problem to determining the sign at one point. At $x = 1000$ the expression is positive. The immediate result is that the domain for which the above expression is positive is the union of regions VI, IV, III, and I.

Not only is this technique less involved, it actually, by its nature, enhances the typical student's insight into the problem. The student has increased benefit and decreased work.

The benefits are even clearer in this second example. Suppose we require the domain for which $(x+\sqrt{5})(x-\sqrt{5})/(x+\sqrt{6})(x-\sqrt{6})$ is positive. Here choosing a test point is very difficult. However, the technique presented here handles this example easily.

See also William E. Allen, Graphing polynomials by using the powers of the factors, *Mathematics Teacher* 77 (1984) 109–110.

Lincoln for President

"Fellow lawyers observed that desire for understanding had become almost a passion with Lincoln. He was never satisfied in handling a thought until, as he said, he had bounded it east, west, north, and south and could express it in the simplest clearest language. He had always had a bent for mathematics, and now as a mental discipline, he mastered the first six books of Euclid, conning their pages in the cheerless hotel rooms or on the lonely prairie as he lolled in his buggy while his horse plodded on unguided from town to town."

Benjamin P Thomas, *Abraham Lincoln*, Knopf, 1952.
Submitted by E. M. Klein, Whitewater, WI.