Some Extensions of a Ubiquitous Geometric Limit Problem

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Anyone teaching calculus today has almost certainly come across the following geometric limit, which I first observed in a textbook by Sherman K. Stein [*Calculus and Analytic Geometry,* 4th ed., McGraw-Hill, New York, 1987, p. 47]. It is found in many other current textbooks as well.

> Consider the sector of a unit circle with angle θ radians as shown in Figure 1. Let A_T be the area of triangle ABC, while A_C is the area of the curved shape ABC. What is the limit of the ratio A_T/A_C as the angle θ approaches 0?

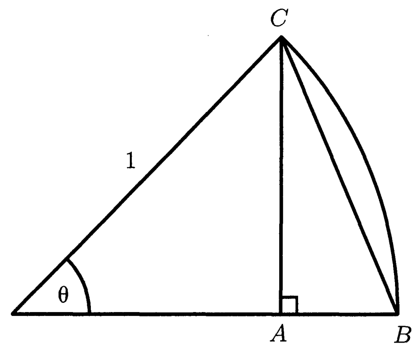

Figure 1

Typically, the student is first urged to guess (because the answer *is* surprising) and then directed to evaluate the limit using l'Hôpital's rule. The calculation yields

$$\lim_{\theta\to\infty} \frac{\frac{1}{2}(\sin\theta - \sin\theta\,\cos\theta)}{\frac{1}{2}(\theta - \sin\theta\,\cos\theta)} = \frac{3}{4}.$$

This problem provides many extensions for students to explore. For example, what happens if the circle is replaced by other conics? By other curves?

In order to see the framework of the problem more clearly, we coordinatize in such a way that the relevant part of the circle (or conic, or other curve) is the graph of a function $y = f(x)$ that passes through the origin and lies in the first quadrant (Figure 2). The graph of $f(x) = (ax^2 + bx)^{1/2}$ with $b \neq 0$ is a portion of a conic section with the x-axis as a line of symmetry (a hyperbola if $a > 0$, ellipse if $a < 0$, and parabola if $a = 0$). More generally, we will consider $f(x) = [p(x)]^r$, where $r > 0$

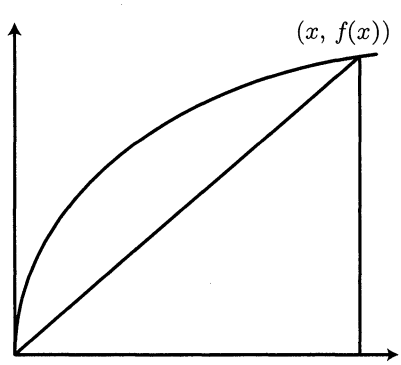

Figure 2

and $p(x)$ is a polynomial with $p(x) > 0$ for all x in some interval $(0, d)$. Other families of functions could, of course, also be considered, but this family yields a particularly nice result. The areas we are concerned with are now given by

$$A_T = \frac{1}{2}xf(x) \qquad \text{and} \qquad A_C = \int_0^x f(t)dt$$

and the limit in question by

$$\lim_{x\to 0}\frac{A_T}{A_C} = \lim_{x\to 0}\frac{\frac{1}{2}xf(x)}{\int_0^x f(t)dt}.$$

Using l'Hôpital's rule and the fundamental theorem of calculus, we have

$$\begin{aligned}\lim_{x\to 0}\frac{A_T}{A_C} &= \frac{1}{2}\lim_{x\to 0}\frac{xf'(x)+f(x)}{f(x)} \\ &= \frac{1}{2}\left[\lim_{x\to 0}\frac{xr[p(x)]^{r-1}p'(x)}{[p(x)]^r}+1\right] \\ &= \frac{1}{2}\left[\lim_{x\to 0}\frac{xrp'(x)}{p(x)}+1\right].\end{aligned}$$

But $\lim_{x\to 0}[xp'(x)]/[p(x)] = k$, where k is the lowest power that appears in the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_k x^k$ ($a_k \neq 0$). The final result, then, is

$$\lim_{x\to 0}\frac{A_T}{A_C} = \frac{1}{2}(rk+1).$$

Thus in the special case of the conics, where $r = 1/2$ and $k = 1$,

$$\lim_{x\to 0}\frac{A_T}{A_C} = \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4},$$

the same value obtained for the circle.

A group project that calculus students might find rewarding is first to compute $\lim_{x\to 0} A_T/A_C$ for various specific functions and then attempt to formulate general results.

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Critical Points of Polynomial Families

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Using computers or graphing calculators opens new opportunities for rich classroom investigative experiences. Problems can be stated, not necessarily in general terms, and students are encouraged to experiment, generate patterns, and explore conjectures. The purpose of this note is to share a problem suitable for a first-year calculus course, regarding critical points of a one-parameter family of polynomials $\{f_t(x)\}$.

Since critical points (i.e., points $(u, f(u))$ where $f'(u) = 0$) are used in determining the extrema of a polynomial $f(x)$, it is interesting to see the effect on these points