if the origin is at the center. By Cavalieri's principle, the area of the stack of n copies of E is equal to the area of E'.

Now stack m copies of Figure 1, one above the other, as shown in Figure 2. Then a vertical line at distance x from the center of E' cuts the m copies of E' in m segments of combined length $2mb\sqrt{1-x^2/(na)^2}$. But this is the length in which the vertical line cuts the *circular disk* E'' bounded by $x^2/(na)^2 + y^2/(mb)^2 = 1$. By Cavalieri's principle the area of the stack of m copies of E' is equal to the area of E'', which is $\pi namb$. This is the combined area of the nm copies of the ellipse E, and so the area of E is πab, as desired.

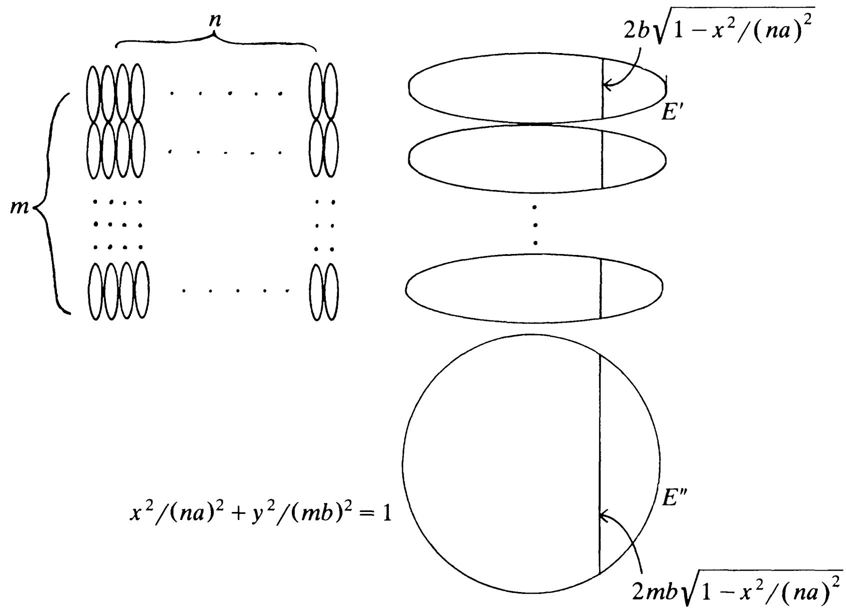

Figure 2

If the ratio a/b is not rational, then ellipse E can be approximated, arbitrarily closely, by ellipses having axes of lengths a and b^* such that a/b^* is rational. Thus, we can show that the area of E is πab for all a and b.

The Isoperimetric Quotient: Another Look at an Old Favorite

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Kouba [1] considered that calculus problem where one cuts a wire into two pieces and forms two figures, each of a prescribed shape, in such a way as to enclose minimum area. It seems worthwhile to pursue this a bit further because of the interesting geometric notions involved. In particular, we shall proceed in the spirit of Niven [2] by dealing with a somewhat more general problem in a manner that avoids calculus and emphasizes the underlying geometry. Perhaps some of these ideas will provide inspiration and motivation for instructors to explore the traditional problems of calculus from more than one viewpoint in an effort to broaden the horizons of their students.

We begin with the observation that associated with a plane figure of perimeter p and area A is the *isoperimetric quotient* p^2/A, a quantity that is of course independent of size, having the same value for similar figures. Thus, for example, all circles have isoperimetric quotient 4π, while all squares have isoperimetric quotient 16. The famous isoperimetric theorem asserts that of all simple closed curves in the plane having the same perimeter, the circle encloses the maximum area. This is embodied in the isoperimetric inequality, which is just the statement that the isoperimetric quotient of any simple closed plane curve is at least that of a circle, this is, $p^2/A \geq 4\pi$ (see Niven [**2**, Chap. 4, 12]). The fact that of all plane quadrilaterals of the same perimeter the square encloses maximum area is equivalent to the inequality $p^2/A \geq 16$ for any quadrilateral of perimeter p and area A (see [**2**, § 3.3]).

Suppose now that $F_1, \dots, F_n$ are given plane figures with respective isoperimetric quotients $\lambda_1, \dots, \lambda_n$ (here, for convenience of later exposition, we differ from Kouba [**1**] in emphasizing λ_i, which is the reciprocal of what he denotes by k_i). We are given a wire of length p and asked to cut it into n pieces, using these to form similar copies of $F_1, \dots, F_n$ in such a way as to enclose the minimum total area. If p_i is the length of a piece of the wire forming the perimeter of a figure similar to F_i, then, since $p_i^2/A_i = \lambda_i$, the area enclosed is $A_i = p_i^2/\lambda_i$. Thus the sum of the areas enclosed is

$$A = A_1 + \cdots + A_n = p_1^2/\lambda_1 + \cdots + p_n^2/\lambda_n,$$

the quantity we want to minimize, where $p_1 + \cdots + p_n = p$, the length of the given wire.

From the strict convexity of the function $f(x) = x^2$, we have

$$\mu_1 x_1^2 + \cdots + \mu_n x_n^2 \geq (\mu_1 x_1 + \cdots + \mu_n x_n)^2$$

for all $x_1, \dots, x_n$ and any $\mu_1, \dots, \mu_n > 0$ such that $\mu_1 + \cdots + \mu_n = 1$. Equality holds precisely when $x_i = x_j$ for all i, j. If we set $\mu_i = \lambda_i/(\lambda_1 + \cdots + \lambda_n)$ and $x_i = p_i/\lambda_i$ in the preceding inequality, we obtain

$$\sum \left[\frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n} \left(\frac{p_i}{\lambda_i} \right)^2 \right] \geq \left[\sum \left(\frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n} \cdot \frac{p_i}{\lambda_i} \right) \right]^2,$$

or, on simplification,

$$p_1^2/\lambda_1 + \cdots + p_n^2/\lambda_n \geq \frac{(p_1 + \cdots + p_n)^2}{\lambda_1 + \cdots + \lambda_n} = \frac{p^2}{\lambda_1 + \cdots + \lambda_n},$$

with equality holding if and only if $p_i/\lambda_i = p_j/\lambda_j = p/(\lambda_1 + \cdots + \lambda_n)$ for all i, j. So in our problem we find that the sum of the areas $A_1 + \cdots + A_n$ is always at least $p^2/(\lambda_1 + \cdots + \lambda_n)$, and attains this minimum value when the lengths of the pieces are apportioned according to

$$p_i = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n} p.$$

This gives, when $n = 2$, the result derived in [**1**].

If we approach the general problem from the viewpoint of the calculus, we are confronted with the multivariable problem of minimizing the function

$f(p_1,\dots,p_n)=p_1^2/\lambda_1+\cdots+p_n^2/\lambda_n$ over the set of $(p_1,\dots,p_n)$ satisfying the constraint $g(p_1,\dots,p_n)=p_1+\cdots+p_n=p$, a given constant. The usual method of Lagrange multipliers applies here, and it is illuminating to view the solution in its equivalent form of "tangent level sets," expressing the fact that at a point $(p_1,\dots,p_n)$ where an extremum is attained the gradient vector of f is parallel to the gradient vector of g. This gives once more the condition $p_i/\lambda_i=p_j/\lambda_j$ for all $i,j=1,\dots,n$. Note that this point $(p_1,\dots,p_n)$ is where the tangent hyperplane of the ellipsoid $p_1^2/\lambda_1+\cdots+p_n^2/\lambda_n=p^2/(\lambda_1+\cdots+\lambda_n)$ has equation $p_1+\cdots+p_n=p$, making equal angles with the coordinate axes.

As a final remark, we note that analogous problems in higher dimensions are easily solved by the same approach. For example, utilizing the convexity of the function $f(x)=x^{3/2}$, the reader may verify that n three-dimensional solids of prescribed shapes and respective surface areas $S_1,\dots,S_n$, with $S_1+\cdots+S_n=S$, a given constant, have minimum total volume when

$$S_i=\frac{\lambda_i}{\lambda_1+\cdots+\lambda_n}S,\quad i=1,\dots,n,$$

where λ_i is the analogue of the isoperimetric quotient, defined by the similarity invariant $\lambda_i=S_i^3/V_i^2$, for a solid with surface area S_i and volume V_i.

References

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A Productive Error in a Trigonometry Text

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Textbook misprints and ill-posed problems can be sources of frustration for students and authors alike. Sometimes, however, they may actually enhance learning by encouraging deeper exploration of concepts and discovery of relationships that might otherwise be overlooked. In this capsule we consider a trigonometry exercise which has appeared in several editions of a popular textbook series, despite the fact that it is ill-posed and the answer given in the text is not feasible. It is apparently intended as a routine exercise using the law of cosines, but we shall see that it can be used for other instructional purposes as well.

> *Two ships S and T are visible from an airplane A flying at an altitude of* 10,000 *feet* (*see Figure* 1). *If the angles of depression from A to S and T are* 37° *and* 21°, *respectively, and* $\angle SAT$ *is* 130°, *determine the distance between the ships.*

The author's intent seems clear. Using right triangle trigonometry on both $\triangle AWS$ and $\triangle AWT$ yields $\overline{AS}\approx 16{,}616$ feet and $\overline{AT}\approx 27{,}904$ feet, respectively. Applying the law of cosines to $\triangle SAT$ then gives $\overline{ST}\approx 40{,}630$ feet, the answer in the text (using four significant figures rather than five in the computations has little effect on the value obtained for $\overline{ST}$). Before we show that $\overline{ST}$ cannot possibly have this value, we mention two features of the exercise that seem to cause trouble for some students.