CLASSROOM CAPSULES

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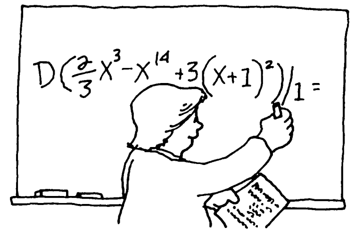

A Generalization of the Mean Value Theorem for Integrals

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Consider the special class of Riemann sums for a continuous function f over $[a, b]$, in which the "sample points" c_i divide the corresponding subintervals in a fixed proportion. In other words, for a partition $a = x_0, x_1, \dots, x_n = b$ there is a fixed number t in $[0,1]$ such that $c_i = x_{i-1} + t\,\Delta x_i$, $i = 1, 2, \dots, n$, where $\Delta x_i = x_i - x_{i-1}$. The Riemann sums are then

$$r_n(t) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f(x_{i-1} + t\,\Delta x_i)\,\Delta x_i.$$

Thus $r_n(0)$, $r_n(\frac{1}{2})$, $r_n(1)$ are the left endpoint, midpoint, and right endpoint Riemann sums for the partition.

Question. For a given partition does there exist a c in $(0,1)$ such that $r_n(c) = \int_a^b f(x)\,dx$?

The answer is yes! For $n = 1$ this is just the mean value theorem for integrals, and the general result easily reduces to this special case.

Proof. The function $r_n(t)$ is continuous on $[0,1]$ since the functions $f(x_{i-1} + t\,\Delta x_i)$ are composites of the continuous function f with the continuous (linear) functions $x_{i-1} + t\,\Delta x_i$. Continuity implies integrability, and

$$\int_0^1 r_n(t)\,dt = \int_0^1 \left[\sum_{i=1}^{n} f(x_{i-1} + t\,\Delta x_i)\,\Delta x_i\right] dt = \sum_{i=1}^{n} \int_0^1 f(x_{i-1} + t\Delta x_i)\,\Delta x_i\,dt$$
$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(u_i)\,du_i = \int_a^b f(x)\,dx,$$

where $u_i = x_{i-1} + t\,\Delta x_i$. Thus if $I = \int_a^b f(x)\,dx$, then $\int_0^1 (r_n(t) - I)\,dt = 0$, and the mean value theorem for integrals yields a c between 0 and 1 with the property that $r_n(c) - I = 0$. This proves the assertion. ∎

It is interesting that the terms in the sequence $\{r_n(t)\}$ are all continuous functions whose integrals over $[0,1]$ equal I, and from above their graphs all intersect the horizontal line $y = I$. Moreover, provided that the mesh of the partitions $a = x_0, x_1, \ldots, x_n = b$ approaches zero as $n \to \infty$, the sequence $\{r_n(t)\}$ converges *uniformly* on $[0,1]$ to the constant function I, since all Riemann sums of f relative to partitions with sufficiently small mesh differ from I by as little as we please. Figure 1, produced using *Mathematica*, shows the graphs of the constant function $I = \int_1^4 (e^x/x)\,dx \approx 17.736$ and the functions $r_{10}(t), r_{20}(t), r_{30}(t)$ on $[0,1]$, using partitions of $[1,4]$ into equal subintervals. The features just mentioned of the sequence $\{r_n(t)\}$ are apparent in the figure. Do you see from the graph that the midpoint sums give a good estimate of this integral?

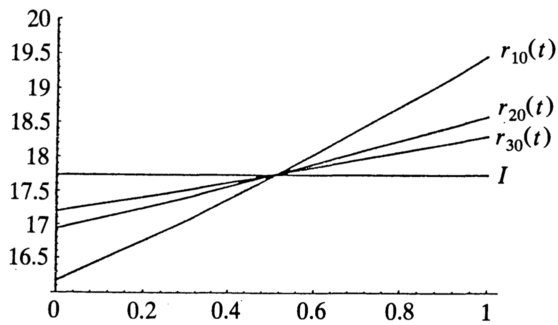

Figure 1

Card Shuffling in Discrete Mathematics

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The problem of determining the number of perfect shuffles required to return a deck of cards to its original order is rich with ideas from elementary algebra, discrete mathematics, and number theory [**1**]. Herstein and Kaplansky [**2**] used card shuffling to motivate a discussion of permutation groups. We have found that for beginning students in discrete mathematics, card shuffling can bring to life the abstract notions of relations and digraphs.

If the overhand shuffle (described below) is applied repeatedly to a deck with an even number of cards, eventually the deck will return to its original order. Let $f(n)$ denote this fundamental period of the shuffling process applied to a deck of $2n$ cards. Although a closed formula for $f(n)$ has yet to be found, we shall see that $f(n)$ is easily computed because it has a simple algebraic interpretation. For more advanced students in a number theory or abstract algebra course, the computation of $f(n)$ is seen to involve the structure of the multiplicative group of units in the ring of integers modulo $2n - 1$.

The overhand shuffle. Cut the deck of $2n$ cards (Figure 1a) into two equal stacks, say with the upper half placed on the right as in Figure 1b. Beginning with the stack on the right, the top card is alternately drawn from each pile and placed in a third pile (Figure 1c) until the two stacks are exhausted. Note that this shuffle interchanges the top and bottom cards of the deck, so clearly $f(n)$ is always even.