Finally, suppose X and X^* are independent and identically distributed random variables having mean 0 and variance 1, and let $Y = X^* - \varepsilon X$. Then

$$\rho(X, Y) = \frac{-\varepsilon}{\sqrt{1 + \varepsilon^2}}$$

and

$$E(Y|X) = E(X^*|X) - \varepsilon E(X|X) = -\varepsilon X.$$

Thus, it is possible for $E(Y|X = x)$ to be a strictly decreasing function of x and yet have $\rho(X, Y)$ arbitrarily close to 0.

Another Proof of Jensen's Inequality

Norman Schaumberger and Bert Kabak, Bronx Community College, Bronx, NY

In this capsule we use the derivative to prove that if $f(x)$ is concave down on the interval $a < x < b$ and if $a < x_i < b$, $i = 1, 2, \ldots, n$, then

$$\frac{1}{n}\sum_1^n f(x_i) \le f\left(\frac{\sum_1^n x_i}{n}\right). \tag{1}$$

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$. This is known as Jensen's inequality and the usual proof uses an adaptation of Cauchy's proof of the arithmetic-geometric mean inequality. (See, for example, I. Niven, *Maxima and Minima Without Calculus*, MAA, 1981, p. 87.) Our argument rests on the following proposition.

If $a < x < b$ and $A = (1/n)\sum_1^n x_i$ where $a < x_i < b$, then

$$f(x) - xf'(A) \le f(A) - Af'(A), \tag{2}$$

with equality if and only if $x = A$.

Since $f(x)$ is concave down on (a, b), its second derivative is negative in this interval. So, (2) follows from the observation that $g(x) = f(x) - xf'(A)$ takes its maximum in (a, b) at $x = A$, because its derivative $g'(x) = f'(x) - f'(A)$ is monotone decreasing on this interval and thus vanishes if and only if $x = A$. Applying (2) to each x_i and adding, we obtain

$$\sum_1^n f(x_i) - f'(A)\sum_1^n x_i \le nf(A) - nAf'(A)$$

or

$$\sum_1^n f(x_i) \le nf\left(\frac{\sum_1^n x_i}{n}\right).$$

Furthermore, there is equality if and only if each x_i in (2) equals A; that is, if and only if $x_1 = x_2 = \cdots = x_n$.

Two familiar trigonometric inequalities are immediate consequences of (1). If $f(x) = \sin x$ then $f''(x) = -\sin x < 0$ for $0 < x < \pi$. It follows that

$$\sum_1^n \sin x_i \leq n \sin\left(\frac{\sum_1^n x_i}{n}\right) \quad \text{for } 0 < x_i < \pi, \quad i = 1, 2, \ldots, n,$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

If $f(x) = \cos x$ then $f''(x) = -\cos x < 0$ for $-\pi/2 < x < \pi/2$. Thus

$$\sum_1^n \cos x_i \leq n \cos\left(\frac{\sum_1^n x_i}{n}\right) \quad \text{for } -\frac{\pi}{2} < x_i < \frac{\pi}{2}, \quad i = 1, 2, \ldots, n,$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Jensen's inequality also gives an immediate proof of the arithmetic-geometric mean inequality. Putting $f(x) = \ln x$, $f''(x) = -1/x^2 < 0$ for $x > 0$. Hence

$$\sum_1^n \ln x_i \leq n \ln\left(\frac{\sum_1^n x_i}{n}\right) \quad \text{for } x_i > 0, \quad i = 1, 2, \ldots, n,$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

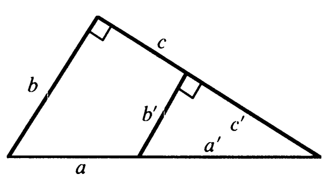
Finally we note that if $f''(x) > 0$ then the inequality in (1) is reversed. Thus, for example, if $f(x) = \tan x$, $f''(x) = 2\sec^2 x \tan x > 0$ for $0 < x < \pi/2$ and it follows that

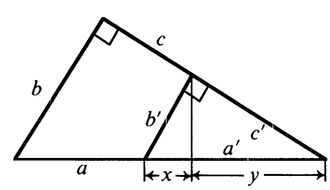
$$\sum_1^n \tan x_i \geq n \tan\left(\frac{\sum_1^n x_i}{n}\right) \quad \text{for } 0 < x_i < \frac{\pi}{2}, \quad i = 1, 2, \ldots, n$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Pythagorean Theorem: $a \cdot a' + b \cdot b' = c \cdot c'$

Enzo R. Gentile, Buenos Aires, Argentina





Proof. By similarity

$$\frac{x}{b} = \frac{b'}{a} \quad \text{or} \quad a \cdot x = b \cdot b' \quad \text{and} \quad \frac{y}{c} = \frac{c'}{a} \quad \text{or} \quad a \cdot y = c \cdot c'.$$

Therefore, $a \cdot a' = a \cdot (x + y) = b \cdot b' + c \cdot c'$.