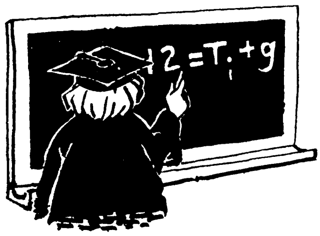

CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

Readers are invited to submit material for consideration to:

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Inverse Hyperbolic Functions as Areas

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It was just one of those things. In a book which I had for so many years accidentally opened to page 193, while thinking about something else, I automatically began to read the right column: "... inverse hyperbolic functions. The inverses of the hyperbolic functions; written $\sinh^{-1}z$, $\cosh^{-1}z$, etc., and read inverse hyperbolic sine of z, etc. Also called archyperbolic functions." Here I dropped what was on my mind and read it again. The book was *Mathematics Dictionary* by James and James. Of course, that must be a mistake. There is no such thing as archyperbolic functions. But there are area functions, which are inverses of hyperbolic functions. Why are they called area functions? Is there a connection between the area and the inverse hyperbolic functions? Well, let's see.

Is $(\sin x)(\cos^{-1}x) = \tan x$? Believe it or not, nobody knows the answer! The reason is simply the lack of standards in mathematical notation. Some of my colleagues use $\cos^{-1}x$ to denote $1/\cos x$, and some use it to denote $\operatorname{arc}\cos x$.

To avoid confusion, most of my colleages in Europe use arc functions as the inverse of trigonometric functions, and area functions as the inverse of hyperbolic functions: instead of $\sinh^{-1}x$ they write $\operatorname{ar}\sinh x$; instead of $\cosh^{-1}x$ they write $\operatorname{ar}\cosh x$, etc.

Why area functions? To answer this, first recall that the area of a sector of a curve is given in polar coordinates by

$$A = \tfrac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta. \tag{1}$$

This can be changed into rectangular coordinates via

$$r^2 = x^2 + y^2, \qquad \theta = \arctan \frac{y}{x}.$$

Since

$$d\theta = \left(\frac{1}{1 + y^2/x^2} \right) \left(\frac{x\,dy - y\,dx}{x^2} \right) = \frac{x\,dy - y\,dx}{x^2 + y^2},$$

(1) becomes

$$A = \tfrac{1}{2} \int_C x\,dy - y\,dx. \tag{2}$$

Now consider the parametric equations

$$x = \cosh t \qquad y = \sinh t \qquad (-\infty < t < \infty).$$

By squaring and subtracting these two equations, and recalling that $\cosh^2 t - \sinh^2 t = 1$, we obtain the equation of an equilateral hyperbola

$$x^2 - y^2 = 1.$$

Let's find the area of the sector of this hyperbola determined by the parameter from 0 to t. Since

$$dx = \sinh t\,dt \quad \text{and} \quad dy = \cosh t\,dt,$$

(2) yields

$$A = \tfrac{1}{2} \int_0^t (\cosh^2 t - \sinh^2 t)\,dt = \frac{t}{2}.$$

From $t = 2A$, we see that the parameter t of the hyperbolic functions is numerically equal to twice the area of the sector of an equilateral hyperbola.

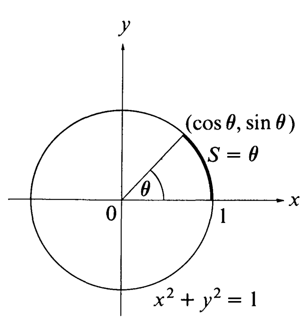

Figure 1.

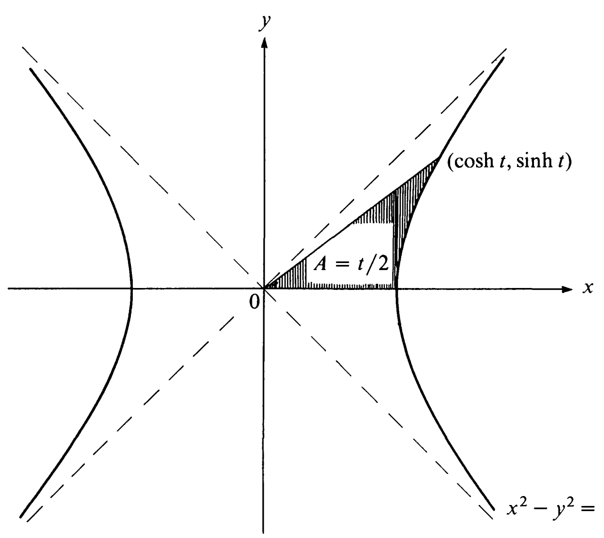

Figure 2.

Thus, $\cosh^{-1}x$ and $\sinh^{-1}x$ are related to the (parametrically determined) hyperbola's sector area A just as $\cos^{-1}x$ and $\sin^{-1}x$ are related to the (parametrically determined) circle's arc length S.

Based on these parametric interpretations, inverse trigonometric functions are called arc functions, and inverse hyperbolic functions are called area functions.

A Self-contained Derivation of the Formula $\frac{d}{dx}(x^r) = rx^{r-1}$ for Rational r

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To establish that $\frac{d}{dx}(x^r) = rx^{r-1}$ for rational r, calculus texts invoke earlier rules of differentiation (quotient rule, chain rule with implicit differentiation, etc.) to first prove this for special cases of r. The purpose of this note is to show that it is possible to establish this result directly, without having to resort to earlier theorems on differentiation. This proof can serve as an instructive exercise for capable students.

Let $r = m/n$, where m and n are positive integers. Then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{m/n} - (x)^{m/n}}{h} = \lim_{h\to 0} \frac{\left[\left\{(x+h)^{1/n}\right\}^m - \left\{x^{1/n}\right\}^m\right]}{\left[\left\{(x+h)^{1/n}\right\}^n - \left\{x^{1/n}\right\}^n\right]}. \tag{1}$$

By letting $a = (x+h)^{1/n}$ and $b = x^{1/n}$ in the difference formula

$$a^N - b^N = (a-b)(a^{N-1} + a^{N-2}b + \cdots + ab^{N-2} + b^{N-1}),$$

and separately considering $N = m$ and $N = n$, we see that (1) becomes

$$f'(x) = \lim_{h\to 0} \frac{\left\{(x+h)^{1/n} - x^{1/n}\right\} \sum_{i=1}^{m} \left\{(x+h)^{1/n}\right\}^{m-i} \left\{x^{1/n}\right\}^{i-1}}{\left\{(x+h)^{1/n} - x^{1/n}\right\} \sum_{i=1}^{n} \left\{(x+h)^{1/n}\right\}^{n-i} \left\{x^{1/n}\right\}^{i-1}}$$

$$= \frac{\sum_{i=1}^{m} x^{(m-i)/n} \cdot x^{(i-1)/n}}{\sum_{i=1}^{n} x^{(n-i)/n} \cdot x^{(i-1)/n}}.$$

In particular,

$$f'(x) = \frac{\sum_{i=1}^{m} x^{(m-1)/n}}{\sum_{i=1}^{n} x^{(n-1)/n}} = \frac{mx^{(m-1)/n}}{nx^{(n-1)/n}} = (m/n)x^{(m/n)-1}. \tag{2}$$