and summing over $d_1, d_2 \in \{0, 1, 2, \ldots, 9\}$ yields

$$P\left[\text{The 3rd digit of } \sqrt{U} = d_3\right] = \sum_{d_2=0}^{9}\left(\sum_{d_1=0}^{9} \frac{200d_1 + 20d_2 + 2d_3 + 1}{10^6}\right)$$

$$= \sum_{d_2=0}^{9} \frac{20d_2 + 2d_3 + 901}{10^5}$$

$$= \frac{2d_3 + 991}{10^4}.$$

Our result for $j = 2$ is in agreement with formula (4) (with $b = 0$) of Schmidt and Lacher's article "Probabilistic Repeatability Among Some Irrationals" [CMJ 15 (September 1984) 330–332].

Example 3. Consider $\alpha = 1/2$ and $j = 2$. If $d_1 \in \{0, 1, 2, \ldots, 9\}$, then (4) yields

$$P\left[\text{The 2nd digit of } \sqrt{U} = d_2\right] = \frac{2d_2 + 91}{10^3} \qquad \text{for} \quad d_2 \in \{0, 1, 2, \ldots, 9\}.$$

If $d_1 \in \{1, 2, \ldots, 9\}$ only, then by (2)

$$P\left[\text{The 2nd digit of } \sqrt{U} = d_2\right] = \frac{2d_2 + 101}{(11)(100)} \qquad \text{for} \quad d_2 \in \{0, 1, 2, \ldots, 9\}.$$

The results stated in (4) and (2) are different because we allow $d_1 = 0$ in (4), but require $d_1 \neq 0$ in (2). This means that l can take 10^j values in (4), whereas k can take $9(10^{j-1})$ values in (2). Since the joint distributions of $(d_1, d_2, \ldots, d_j)$ are different in (2), the marginal distributions of d_j will also be different in the two models.

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Irrationality Made Easy

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In his May, 1984, lecture to the Metropolitan New York Section of the MAA, Professor Ivan Niven reminded us of the following proof that $\sqrt{2}$ is irrational. If $\sqrt{2}$ is rational, then there is a smallest positive integer b such that $b\sqrt{2}$ is an integer. But $b\sqrt{2} - b$ is a smaller positive integer and $(b\sqrt{2} - b)\sqrt{2}$ is an integer; so we have a contradiction. It is not hard to extend this proof to arbitrary roots of arbitrary positive integers.

If n and k are integers greater than one, and n is not the kth power of an integer, then $n^{1/k}$ is irrational.

Suppose $n^{1/k}$ is rational. Then $n^{i/k}$ is rational for each positive integer i, and so there exists a sequence of positive integers a_i such that each $a_i n^{i/k}$ is an integer. Then $a = a_1 a_2 \cdots a_{k-1}$ is a positive integer, and $an^{i/k}$ is integral for $1 \leqslant i \leqslant k-1$. Let b be the smallest positive integer such that $bn^{i/k}$ is an integer for $1 \leqslant i \leqslant k-1$. Since n is not a kth power, there is an integer m such that $m < n^{1/k} < m+1$. Let $c = bn^{1/k} - bm$. Then c is a positive integer less than b. But for $1 \leqslant i \leqslant k-2$,

$$cn^{i/k} = bn^{(i+1)/k} - bn^{i/k}m$$

is an integer. And for $i = k-1$,

$$cn^{(k-1)/k} = bn - bn^{(k-1)/k}m$$

is an integer. This contradicts the definition of b, and completes the proof.

The proofs usually given for this result make some use of the Fundamental Theorem of Arithmetic, which may require too much time to explain (let alone to prove) in, say, a precalculus course. As Professor Niven observed, the fact that this proof assumes the well-ordering principle has never been seen to bother a student.

The Derivatives of Arcsec x, Arctan x, and Tan x

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Beginning calculus students know that the shaded sector in Figure 1 has area equal to $(1/2)r^2\theta$. Here we show how different expressions for θ can yield formulas for the above titled derivatives.

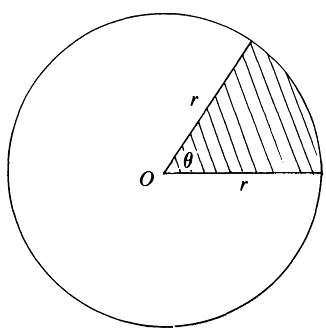

Figure 1.

To obtain the formula for $d(\text{arcsec}\, x)/dx$, let (Figure 2) BE and CD be arcs of circles with center O and radii x and $x + \Delta x$, respectively. Then,

$$\text{area(sector } OBE) < \text{area(triangle } OBD) < \text{area(sector } OCD). \qquad (*)$$